Advanced Global Analysis I
Exercise series 7

January 8, 2016
Due: January 15, 2016

Exercise 25. (4 points)
For $x \in \mathbb{R}^n \setminus \{0\}$ let

$$G(x) \begin{cases} \frac{1}{2\pi} \log \| x \|, \ n = 2 \\ -\frac{1}{(n-2)\omega_n \|x\|^{n-2}}, \ n \geq 3, \end{cases}$$

where $\omega_n = \mathrm{vol}(S^{n-1})$. Then $G(x)$ is locally integrable. Let δ be the δ- distribution. Show that

$$\Delta G = \delta.$$

(G is called fundamental solution).

Exercise 26. (2 points)
Let (X, g) be a compact Riemannian manifold. Let

$$\Delta \colon C^\infty(X) \to C^\infty(X)$$

be the Laplace operator.
Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Show that for $k > n(n+1)$, $(\Delta - \lambda)^{-k}$ is an integral operator with a continuous kernel $K(x, y)$.
Describe the kernel explicitly in terms of the spectral resolution of Δ.

Exercise 27. (4 points)
Let $v_1, ..., v_n \in \mathbb{R}^n$ be a basis.
Let

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n.$$

Then $\Lambda \subset \mathbb{R}^n$ is a lattice. Let

$$T \colon = \mathbb{R}^n / \Lambda.$$

Then T is an n-dimensional torus. Equip T with the metric induced from $\mathbb{R}^n$ and let Δ be the corresponding Laplace operator.
Let

$$\Lambda^\star = \{\mu \in \mathbb{R}^n \colon \langle \mu, \lambda \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\}$$

be the dual lattice.
For $\mu \in \Lambda^\star$ let

$$\varphi_\mu(x) = \frac{e^{2\pi i \langle}\mu, x\rangle}{\sqrt{\mathrm{vol}(T)}}.$$

Show that $\{\varphi_\mu\}_{\mu \in \Lambda^\star}$ is an orthonormal basis of $L^2(T)$ consisting of eigenfunctions of Δ.

Exercise 28. (2 points)
Let (X, g) be a compact Riemannian manifold with nonempty boundary ∂X. Let Δ be the Laplace operator of X. Consider Δ as operator in $L^2(X)$ with domain $C_c^\infty(X)$:

Let

$$\Delta_D\colon \operatorname{dom}(\Delta_D) \to L^2(X)$$

be the selfadjoint extension of Δ with respect to Dirichlet boundary conditions. Then

$$\operatorname{dom}(\Delta_D) = H^2(X) \cap H_0^1(X)$$

where

$$H_0^1(X) = \{f \in H^1(X)\colon f|_{\partial X} = 0\}.$$

Show that $\Delta_D > 0$.