Advanced Global Analysis I

Exercise series 6

December 18, 2015

Due: January 8, 2016

Exercise 21. (2 points)
Let $P \in \Psi DO_m(\mathbb{R}^n)$. Assume that P is elliptic in the sense that

$$\exists M > 0, A > 0 \; \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\colon \; \| \xi \| > M \Rightarrow |a(x,\xi)| \geq A \| \xi \|^m .$$

Construct $Q_0 \in \Psi DO_{-m}(\mathbb{R}^n)$ such that

$$PQ_0 = \operatorname{Id} + E_1, \; Q_0 P = \operatorname{Id} + E_2,$$

where $E_1, E_2 \in \Psi DO_{-1}(\mathbb{R}^n)$.

Exercise 22. (2 points)
Let $\mathscr{H}$ be a Hilbert space and $T\colon \mathscr{H} \to \mathscr{H}$ a bounded linear operator. Show that the following conditions are equivalent:

1) T is a Fredholm operator.

2) $\dim\ker T, \dim\ker T^\star < \infty$ and T is bounded from below, i.e. there exists $\varepsilon > 0$ such that
$$\forall x \in (\ker T)^\perp\colon \; \|T(x)\| \geq \varepsilon \|x\|.$$

3) T^*T is a Fredholm operator.

Exercise 23. (3 points)
Let $a, b \in \mathbb{R}$, $a < b$. Let $D = i\frac{d}{dx}$. Consider D as operator in $L^2(a,b)$ with domain $\mathscr{D}(D) = C_c^\infty(a,b)$. Let $\tilde{D}$ be a selfadjoint extension of D with domain $\mathscr{D}(\tilde{D}) \supset \mathscr{D}(D)$. Show that there exists $\lambda \in S^1$ such that

$$\mathscr{D}(\tilde{D}) = \{ f \in H^1(a,b)\colon f(b) = \lambda f(a) \}$$

Exercise 24. (3 points)
Let $\mathscr{H}$ be a Hilbert space, $\mathscr{D} \subset \mathscr{H}$ a dense subspace and

$$A\colon \mathscr{D} \to \mathscr{H}$$

a linear operator. Recall that A is called symmetric if

$$\forall x, y \in \mathscr{D}\colon \langle Ax, y \rangle = \langle x, Ay \rangle.$$

Prove the following statements:

a) If A is symmetric, then A is closable. Let $\bar{A}$ be the closure of A. Then $\bar{A}$ is also symmetric.

   Hint: Show that A^* is a closed operator and $A \subset A^*$.

b) Assume that A is symmetric. Let $B \supset A$ be a symmetric extension of A. Then $B \subset A^\star$.

c) Let A be selfadjoint. Then A is closed and A has no nontrivial symmetric extensions.