Advanced Global Analysis I
Exercise series 5

December 11, 2015
Due: December 18, 2015

Exercise 17. (2 points)
Let

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplace operator in $\mathbb{R}^n$. Show that there exist $Q \in \Psi DO_{-2}(\mathbb{R}^n)$ and smoothing operators S_1, S_2 such that

$$\Delta \circ Q = \operatorname{Id} + S_1, \quad Q \circ \Delta = \operatorname{Id} + S_2.$$

Give an explicit description of Q, S_1 and S_2.

Exercise 18. (2 points)
Let $\mathscr{H}$ be a separable Hilbert space. Assume that the unit ball $B \subset \mathscr{H}$ is compact. Show that $\mathscr{H}$ is finite dimensional.

Exercise 19. (3 points)
Let X be a compact Riemannian manifold. Let $A\colon L^2(X) \to L^2(X)$ be a symmetric integral operator with a C^1-kernel, i.e., there exists $K \in C^1(X \times X)$ such that

$$Af(x) = \int_X K(x,y) f(y)\, dy, \quad f \in L^2(X).$$

Show that A is a compact operator.

Exercise 20. (3 points)
Let X be a compact Riemannian manifold. Let $A\colon L^2(X) \to L^2(X)$ be a symmetric integral operator with a C^1-kernel $K(x,y)$. Show that A is a Hilbert-Schmidt operator and the Hilbert-Schmidt norm $\| A \|_2$ of A satisfies

$$\| A \|_2^2 = \int_X \int_X |K(x,y)|^2 dx dy.$$