Advanced Global Analysis I
Exercise series 4

December 4, 2015
Due: December 11, 2015

The Hodge $*$-operator

Let $(V, \langle\cdot,\cdot\rangle)$ be a Eucledean vector space of dimension n. The inner product $\langle\cdot,\cdot\rangle$ induces an inner procuct in $\Lambda^p V$ by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p\rangle := \det\left(\langle v_i, w_j\rangle_{i,j=1}^n\right).$$

Let $e_1, \ldots, e_n \in V$ be an orthonormal basis of V. Let V be oriented by $(e_1, \ldots, e_n)$. The operator $*\colon \Lambda^p V \to \Lambda^{n-p} V$ is defined by

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \varepsilon_{i_1,\ldots,i_p,j_1,\ldots,j_{n-p}} e_{j_1} \wedge \cdots \wedge e_{j_{n-p}},$$

where $\{j_1, ..., j_{n-p}\} = \{1, ..., n\} \setminus \{i_1, ..., i_p\}$ and

$$\varepsilon_{i_1,\ldots,i_p,j_1,\ldots,j_{n-p}} = \operatorname{sign}\begin{pmatrix} 1 \cdots p\,(p+1) \cdots n \\ i_1 \cdots i_p j_1 \cdots j_{n-p}\end{pmatrix}.$$

Exterior and inner multiplication: For $v \in V$ let $\varepsilon\colon \Lambda^* V \to \Lambda^* V$ be defined by

$$\varepsilon(v)\varphi = v \wedge \varphi, \quad \varphi \in \Lambda^* V.$$

Let $i(v)\colon \Lambda^* V \to \Lambda^* V$ be the adjoint of ε, i.e., we have

$$\langle \varepsilon(v)\varphi, \psi\rangle = \langle \varphi, i(v)\psi\rangle, \quad \forall\, \varphi, \psi \in \Lambda^* V.$$

Exercise 13. (2 points) Show that

1. On $\Lambda^p V$ one has $** = (-1)^{p(n-p)}\,\mathrm{Id}$.
2. $\langle v, w\rangle = *(v \wedge *w) = *(w \wedge *v)$, $v, w \in \Lambda^p V$.

Exercise 14. (2 points)
Let (M, g) be an oriented Riemannian manifold. Let $\langle\cdot,\cdot\rangle$ be the inner product in $\Lambda^p(M)$, which is induced by the Riemannian matric. Show that

$$\langle \varphi, \psi\rangle = \int_M \varphi \wedge *\bar{\psi}$$

for all $\varphi, \psi \in \Lambda_c^p(M)$.
Hint: Use Exercise 13, 2), and $*(1) = d\mu_g$ (the volume form).

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Exercise 15. (3 points) Let $d_p^*\colon \Lambda^p(M) \to \Lambda^{p-1}(M)$ be the formal adjoint operator of d_p. Show that

$$d_p^* = (-1)^{n(p+1)+1} * d * .$$

Hint: Use that $\int_M d\omega = 0$ for all $\omega \in \Lambda_c^{n-1}(M)$. (This follows from Stoke's theorem)

Exercise 16. (3 points) For $\xi \in T_x^*(M)$, let $i(\xi)\colon \Lambda^p T_x^*(M) \to \Lambda^{p-1} T_x^*(M)$ denote the interior multiplication by ξ. Show that the principal symbol σ_{d^*} of d^* is given by

$$\sigma_{d^*}(x, \xi) = -i(\xi).$$