# Advanced Global Analysis I
## Exercise series 3

December 4, 2015

Due: December 11, 2015

**Exercise 9.** (3 points) Let $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Assume that $p$ satisfies the following conditions:

1. $\forall \alpha \in \mathbb{N}_0^n$, $\exists\, C_\alpha > 0\colon |D_x^\alpha p(x,\xi)| \leq C_\alpha$ for all $x \in \mathbb{R}^n$.
2. For $\lambda \geq 1$ and $\|\xi\| \geq 1$ one has $p(x, \lambda\xi) = \lambda^m p(x,\xi)$.

Show that $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$.

**Hint:** Use polar coordinates.

**Exercise 10.** (3 points) Let $a \in \widetilde{S}^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Assume that $\operatorname{supp}_{(x,y)} a$ is compact. Let $A\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the induced operator. As we know, $A$ is a peudodifferential operator. Let $p \in S^m$ be the symbol of $A$. Show that

$$p(x,\xi) = e^{-\langle x,\xi\rangle} A(e^{i\langle \cdot,\xi\rangle}).$$

**Schwartz kernel:** Let $T\colon C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ be a linear continous operator. Then there exists a unique distribution $K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\langle Tu, v\rangle = \langle K, v \otimes u\rangle$ for all $u, v \in C_c^\infty(\mathbb{R}^n)$.

**Exercise 11.** (2 points) Let $a \in\in \widetilde{S}^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and let $A$ be the operator defined by $a$. Show that the Schwartz kernel $K_A$ of $A$ is given by

$$K_A = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\, dx.$$

**Exercise 12.** (2 points) Let $A\colon C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator. Then the following statments are equivalent:

1. $A$ is a pseudodifferential operator of order $-\infty$.
2. There exists $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$(Au)(x) = \int_{\mathbb{R}^n} K(x,y) u(y)\, dy.$$