Advanced Global Analysis I
Exercise series 2

November 20, 2015
Due: November 27, 2015

Exercise 5. (2 points) Let $\Omega \subset \mathbb{R}^n$ be open and let

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a differential operator of order m on Ω.

1) Show that there is a unique differential operator P^* of order m on Ω, such that

$$\langle Pf, g\rangle = \langle f, P^* g\rangle$$

for all $f, g \in C_c^\infty(\Omega)$.

2) Let $p(x,\xi)$ be the complete symbol of P. Show that the symbol p^* of P^* is given by

$$p^*(x,\xi) = \sum_{|\alpha| \leq m} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \, \overline{p}(x,\xi).$$

Exercise 6. (2 points) Let $U, V \subset R^n$ be open and $\varphi \colon U \to V$ a diffeomorphism. Define $\varphi^* \colon C^\infty(V) \to C^\infty(U)$ by $\varphi^*(f) = f \circ \varphi$. Let D be a differential operator of order m on U. Define

$$D^\varphi \colon C^\infty(V) \to C^\infty(V)$$

by

$$D^\varphi f = \left(\varphi^{-1}\right)^* D \varphi^* f.$$

Show that D^φ is a differential operator of order m on V. Determine the symbol of D^φ in terms of the symbol of D.

Exercise 7. (3 points) Let $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in $\mathbb{R}^n$. Let $m \in \mathbb{R}^+$. Define $P\colon = (\Delta + \mathrm{Id})^{-m} \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, using the Fourier transform. Show that for every $s \in \mathbb{R}, P$ has a continuous extension

$$P_s \colon H^s(\mathbb{R}^n) \to H^{s+2m}(\mathbb{R}^n).$$

Exercise 8. (3 points) Show that for all $s, s' \in \mathbb{R}, s > s'$, the inclusion

$$H^s(\mathbb{R}^n) \to H^{s'}(\mathbb{R}^n)$$

is not compact.