Advanced Global Analysis I
Exercise series 1

November 13, 2015
Due: November 20, 2015

Exercise 1. (2 points) Let $\alpha, \beta \in \mathbb{N}_0^n$ be multiindeces. Put $\beta \leq \alpha$, if $\beta_i \leq \alpha_i$ for all $i = 1, \ldots, n$. Let $\alpha! := \alpha_1! \cdots \alpha_n!$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}, \quad \beta \leq \alpha.$$

Let $\Omega \subset \mathbb{R}^n$ be open and $f, g \in C^\infty(\Omega)$. Show that

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(D^\beta f\right) \left(D^{\alpha-\beta} g\right).$$

Exercise 2. (2 points) Let

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha \quad \text{and} \quad Q = \sum_{|\beta| \leq m} b_\beta(x) D^\beta.$$

be differential operators in an open subset $\Omega \subset \mathbb{R}^n$ with $a_\alpha, b_\beta \in C^\infty(\Omega)$ for all α and β. Let

$$p(x,\xi) = \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha \quad \text{and} \quad q(x,\xi) = \sum_{|\beta| \leq m} b_\beta(x) \xi^\beta, \quad x \in \Omega,\ \xi \in \mathbb{R}^n,$$

be the symbols of P and Q, respectively. Show that the symbol $r(x,\xi)$ of $P \circ Q$ is given by

$$r(x,\xi) = \sum_{|\gamma| \leq k} \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma p(x,\xi) \cdot D_x^\gamma q(x,\xi).$$

Exercise 3. (3 points) Let M be a manifold and $E, F \to M$ complex vector bundles. Let

$$D \colon \Gamma(M,E) \to \Gamma(M,F)$$

be a differential operator of order $m \in \mathbb{N}_0$. Let $p \in M, \xi \in T_p^\star M, e \in E_p$. Choose $f \in C_c^\infty(M)$, $s \in \Gamma(M,E)$ such that $f(p) = 0$, $df|_p = \xi$, $s(p) = e$. Show that the principal symbol σ_D of D is given by

$$\sigma_D(p,\xi)e = \frac{i^m}{m!} D(f^m s)(p).$$

Exercise 4. (3 points) Compute the principal symbols of the operators $d \colon \Lambda^p(M) \to \Lambda^{p+1}(M)$ and

$$\Delta = d^\star d \colon C^\infty(M) \to C^\infty(M).$$