The Geometry of Shimura Curves and special cycles

Andreas Mihatsch*

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These are notes for the Alpbach pre-workshop in Zürich in April 2015. They contain some more details than the talks themselves.

1 Introduction

The reference for the introduction is [14][Chap. 1]. Let \( B/\mathbb{Q} \) be an indefinite quaternion algebra, i.e. \( B \otimes \mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R}) \). Let

\[
D := D(B) := \prod_{p \text{ s.th. } B \otimes \mathbb{Q}_p \text{Div.alg}} p
\]

denote the discriminant of \( B \). We fix a maximal order \( \mathcal{O}_B \) of \( B \). It is unique up to conjugacy.

Definition 1.1. We define the Shimura curve \( \mathcal{M} \rightarrow \text{Spec} \mathbb{Z} \) associated to \( B \) to be the following stack:

\[
\mathcal{M}(S) = \{(A, \iota) \mid \text{A abelian surface}/S + \iota : \mathcal{O}_B \rightarrow \text{End}(A) + \text{special condition}\}.
\]

Special condition:

\[
\text{charpol}(\iota(b) | \text{Lie } A)(T) = \text{charred}(b)(T) \in \mathcal{O}_S[T] \quad \forall b \in \mathcal{O}_B.
\]

The polynomial on the right is the reduced characteristic polynomial of \( x \). If you are not familiar with the definition of a stack, just think of \( \mathcal{M}(S) \) as the isomorphism classes of pairs \( (A, \iota) \) together with their automorphisms.

Remark 1.2. i) We take it for granted that \( \mathcal{M} \) is a Deligne-Mumford stack, of finite type over \( \mathbb{Z} \).

ii) If \( B = M_2(\mathbb{Q}) \), then we can assume \( \mathcal{O}_B = M_2(\mathbb{Z}) \). Then \( E \mapsto E \times E \) and \( (A, \iota) \mapsto \ker((1)) \) induce an equivalence

\[
\text{Ell}(S) := \{\text{Elliptic Curves}/S\} \cong \mathcal{M}(S).
\]

So \( \mathcal{M} \) is the usual modular curve.

iii) The special condition is closed and automatic in characteristic 0. It ensures flatness of \( \mathcal{M}/\text{Spec } \mathbb{Z} \).

We will prove later in the course:

\*mihatsch@math.uni-bonn.de
Proposition 1.3. The DM-stack $M$ is regular of dimension 2. The morphism $M \to \text{Spec } \mathbb{Z}$ is flat of relative dimension 1, smooth over $\text{Spec } \mathbb{Z}[D(B)^{-1}]$ and has semi-stable reduction at $p | D(B)$. Its fibers are geometrically connected. If $B$ is a division algebra, then $M$ is proper over $\text{Spec } \mathbb{Z}$.

The stack $M$ is a regular surface, so it has a good intersection theory. Also, $M$ is canonically endowed with a whole family of cycles. Let us describe them in the language of Shimura varieties.

1.1 CM cycles on $M$

Let $C := \text{End}_B(B)$ and $G := C^\times$ as algebraic group over $\mathbb{Q}$. Note that $C = B^{op} \cong B$, but it is better to separate these algebras. Let $O_C = \text{End}_{O_B}(O_B)$ which is a maximal order in $C$. It yields $(O_C \otimes \hat{\mathbb{Z}})^\times = : K \subset G(\mathbb{A}_f)$ and there is an isomorphism

$$M_C \cong [G(\mathbb{Q}) \backslash h^\pm \times G(\mathbb{A}_f)/K].$$

Here $G(\mathbb{Q})$ acts on $h^\pm : = \mathbb{C} \setminus \mathbb{R}$ via the choice of an isomorphism $G(\mathbb{R}) \cong GL_2(\mathbb{R})$.

Let $E \subset \mathbb{C}$ be an imaginary quadratic field with embedding $\rho : E \hookrightarrow C$. Let $T := E^\times$ be the corresponding torus with embedding $\rho : T \hookrightarrow G$. This defines a divisor (Heegner divisor)

$$[T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\rho^{-1}(K)] \to M_C$$

on the Shimura curve.

1.2 Special cycles

When trying to extend these cycles to $M/\text{Spec } \mathbb{Z}$, it is better to use a moduli theoretic approach. The special cycles are weighted combinations of integral models of these CM cycles.

Definition 1.4. Let $(A, \iota) \in M(S)$. The space of special endomorphisms of $(A, \iota)$ is the $\mathbb{Z}$-module

$$V(A, \iota) = \{x \in \text{End}_S(A, \iota) \mid \text{tr}(x) = 0\}.$$ 

It is a quadratic space with form

$$-x^2 = Q(x) \cdot \text{id}_A.$$ 

It follows from the classification of $\text{End}(A, \iota) \otimes \mathbb{Q}$ that this form is positive definite.

Definition 1.5. Let $t > 0$ be an integer. Define the special cycle $Z(t) \to M$ as the moduli stack of $(A, \iota, x)$ with $Q(x) = t$.

The forgetful morphism $pr : Z(t) \to M$ is unramified and finite but it is not a closed immersion. The image of $pr$ are the $(A, \iota)$ which admit complex multiplication by $\mathbb{Z}[\sqrt{-t}]$. The complex fiber $Z(t)_C$ is a linear combination of CM cycles as above.
1.3 Relation to modular forms

We assume now that \( B \) is division such that \( M \) is proper. The generic fiber \( Z(t)_C \) is reduced and 0-dimensional, so it is a finite union of points, each with a finite group of automorphisms. Its degree is defined as

\[
\deg Z(t)_C = \sum_{(A,\iota,x)\in Z(t)_C} \frac{1}{|\text{Aut}(A,\iota,x)|}.
\]

The index set of the special cycles is the same as that of elliptic modular forms. We assemble them into a generating series

\[
\phi_{\text{deg}}(\tau) := -\text{vol}(M(\mathbb{C})) + \sum_{t>0} \deg(Z(t)_C) q^t \in \mathbb{C}[[q]].
\]

The motivation for the constant term is that purely formally, \( Z(0) = M \). Also, this volume is known,

\[
\text{vol}(M(\mathbb{C})) = \frac{1}{12} \prod_{p|D(B)} (p-1).
\]

**Proposition 1.6.** The series \( \phi_{\text{deg}} \) is the \( q \)-expansion of a holomorphic modular form of weight \( 3/2 \) and level \( \Gamma_0(4D(B)) \).

The proof of this proposition is not via general arguments. Instead one explicitly computes all degrees \( \deg Z(t) \) and compares them to the Fourier coefficients of a known Eisenstein series! In this course, I will only explain the computation of the degrees.

1.4 Arithmetic special cycles

The reference for Chow groups and intersection theory is [14][Chap. 2].

Note that \( M \to \text{Spec} \mathbb{Z} \) is flat and proper, but \( \text{Spec} \mathbb{Z} \) is not “compact” itself. We use Arakelov theory to define arithmetic cycles \( \mathcal{Z}(t,v) \).

**Definition 1.7.** i) Let \( Z^1_{\mathbb{A}}(M) \) be the group of divisors on \( M \) with coefficients in \( \mathbb{R} \). In other words, \( Z^1_{\mathbb{A}}(M) \) is the \( \mathbb{R} \)-vector space generated by the irreducible closed substacks \( Z \subset M \).

ii) Let \( Z \in Z^1_{\mathbb{A}}(M) \) and denote by \( \delta_Z : C^\infty(M(\mathbb{C})) \to \mathbb{C} \) its \( \delta \)-distribution. A Green function for \( Z \) is a smooth (real-valued) function \( \Xi(v) \) on \( M(\mathbb{C}) \) with logarithmic growth along \( Z_C \) such that the Green equation holds:

\[
d d\ast \Xi(v) + \delta_Z = [\omega]
\]

for some smooth (1,1)-form \( \omega \). We let \( \widehat{Z}^1_{\mathbb{A}}(M) \) denote the \( \mathbb{R} \)-vector space of Arakelov divisors, i.e. of pairs \( (Z,\Xi(v)) \), where \( \Xi(v) \) is a Green function for \( Z \).

iii) To a meromorphic function \( f \in \mathbb{Q}(M)^\times \), we associate the principal Arakelov divisor

\[
\widehat{\text{div}}(f) := (\text{div}(f),-\log |f_C|^2).
\]

iv) The first arithmetic Chow group (with real coefficients) \( \widehat{CH}^1(M) \) is the quotient of \( \widehat{Z}^1_{\mathbb{A}} \) by the \( \mathbb{R} \)-vector space generated by the principal Arakelov divisors.
Remark 1.8. Let $\mathcal{L}$ be a line bundle on $\mathcal{M}$ together with a smooth metric $|\cdot|$ on the complex fiber $\mathcal{L}_C$. Let $s \in \mathcal{Q}(\mathcal{L})^\times$ be a meromorphic section. Then $(\text{div}(s), -\log |s|^2)$ is an Arakelov divisor and every Arakelov divisor with integral coefficients arises in this way.

In his Annals paper, Kudla defines certain functions $\Xi(t,v)$ for $t \in \mathbb{Z}\setminus\{0\}$, which are Green functions for $\mathcal{Z}(t)$ if $t > 0$ and smooth if $t < 0$. For the definition, see [14, Chap. 3.5]. This defines classes in $\hat{\text{CH}}^1(\mathcal{M})$.

$$\hat{Z}(t,v) := \begin{cases} (\mathcal{Z}(t),\Xi(t,v)) & \text{if } t > 0 \\ (0,\Xi(t,v)) & \text{if } t < 0 \\ \text{see KRY (4.2.4)} & \text{if } t = 0. \end{cases}$$

We form the formal generating series

$$\hat{\phi} := \sum_{t \in \mathbb{Z}} \hat{Z}(t,v) q^t \in \hat{CH}^1(\mathcal{M})[[q,q^{-1}]].$$

Then the main result is the following theorem.

Theorem 1.9. For $\tau = u + iv$, the series $\hat{\phi}(\tau)$ is a (nonholomorphic) modular form of weight $3/2$ and level $\Gamma(4D(B))$ with values in $\hat{CH}^1(\mathcal{M})$.

This is Theorem A in [14]. See the introduction there for the precise meaning of this statement.

About the proof: The first Chow group $\hat{CH}^1(\mathcal{M})$ is endowed with an intersection product (see [14][Chap. 2])

$$\langle , \rangle : \hat{CH}^1(\mathcal{M}) \times \hat{CH}^1(\mathcal{M}) \to \mathbb{R}.$$ 

There is a decomposition, orthogonal for $\langle , \rangle$.

$$\hat{CH}^1(\mathcal{M}) = \hat{\text{MW}} \oplus (\mathbb{R}\hat{\omega} \oplus \text{Vert}) \oplus C^\infty(\mathcal{M}(\mathbb{C})).$$

Here, $C^\infty(\mathcal{M}(\mathbb{C}))(0)$ are the functions orthogonal to the constant ones. And Vert is the subspace of $\hat{CH}^1(\mathcal{M})$ generated by divisors $(Y,0)$, where $Y$ is an irreducible component of some fiber of $\mathcal{M}/\mathbb{Z}$. The class $\hat{\omega}$ is the class of the metrized Hodge bundle. The Mordell-Weil space $\hat{\text{MW}}$ is then the orthogonal complement of the other summands. Restriction to the generic fiber defines an isomorphism

$$\hat{\text{MW}} \cong \text{Jac}(\mathcal{M}_Q) \otimes \mathbb{R}.$$ 

Let $\hat{\phi}^\infty$ be the components of $\hat{\phi}$ in $C^\infty(\mathcal{M}(\mathbb{C}))(0)$ and let $\hat{\phi}^0$ be the component in the other summands. For the modularity of $\hat{\phi}^\infty$, see [14, 4.4].

To prove the modularity of $\hat{\phi}^0$, one shows that for every $Z \in \hat{\text{MW}} \oplus (\mathbb{R}\hat{\omega} \oplus \text{Vert})$, the series $\langle \hat{\phi}^0, Z \rangle$ is a modular form. This is done by “explicitly” computing its coefficients $\langle \hat{Z}(t,v), Z \rangle$. These coefficients can then be compared to the coefficients of “known” modular forms, usually Eisenstein series or Theta series. For details, see [14][Chap. 4]. In particular, there are no “abstract” arguments which yield the modularity of $\hat{\phi}$. □
Remark 1.10. The degree of the generic fiber
\[ (Z, \Xi) \mapsto \deg(Z_Q) \]
defines a linear form on \( \widehat{CH}^1(M) \). Thus the modularity of \( \phi_{\deg} \) follows from the modularity of \( \phi \).

1.5 The vertical component

The reference is [14][Chap. 4.3].

In this course, we will focus on the series \( \langle \hat{\phi}, Z \rangle \), where \( Z \in \text{Vert} \) is a vertical divisor.

Recall that \( M \to \text{Spec} \mathbb{Z} \) is a proper flat relative curve with geometrically connected fibers. Let \( \text{Vert} \subset \widehat{CH}^1(M) \) be the subspace generated by the cycles of the form \( (Y, 0) \), where \( Y \) is a linear combination of irreducible components of fibers. The main result of this course is

**Proposition 1.11.** Let \( Z \in \text{Vert} \) a vertical Arakelov divisor. Then
\[ \langle \hat{\phi}, Z \rangle \]
is a (holomorphic) modular form of weight \( 3/2 \).

It suffices to prove this for irreducible divisors \( Z = (Y, 0) \) where \( Y \) is a vertical irreducible component. Let \( p \) be the prime below \( Y \). I did not give the definition of \( \langle \cdot, \cdot \rangle \), it is enough to know for us that
\[ \langle (Z, \Xi), (Y, 0) \rangle = \log(p) \cdot (Z, Y). \]

Here the intersection number \( (Z, Y) \) is defined as usual if \( Z \) and \( Y \) have no common components. The product \( (Y, Y) \) is defined as follows: Write \( V(p) = aY + R \in Z^1_\mathbb{R}(M) \) and define \( (Y, Y) := -\frac{1}{2}(R, Z) \).

Assume that \( p \nmid D(B) \), i.e. assume that \( M \) is smooth over \( p \). We will see:

**Proposition 1.12.** The cycle \( Z(t)[D(B)^{-1}] \) is a divisor. It is flat over \( \text{Spec} \mathbb{Z}[D(B)^{-1}] \).

It follows from the proposition that \( Z(t) \) and \( Y \) have no common components. Thus,
\[ \langle \hat{Z}(t, v), (Y, 0) \rangle = \log(p) \cdot (Z(t), Y) = \begin{cases} 
\log(p) \cdot \deg(Z(t)_C) & \text{if } t > 0 \\
- \log(p) \text{Vol}(M(C)) & \text{if } t = 0 \\
0 & \text{if } t < 0.
\end{cases} \]

Thus \( \langle \hat{\phi}, (Y, 0) \rangle = \log(p) \cdot \phi_{\deg} \). So modularity will follow from that of \( \phi_{\deg} \).

If \( Y \) is an irreducible component over \( p \mid D(B) \), the computation of \( \langle Z(t), Y \rangle \) is more difficult and relies on the \( p \)-adic uniformization of the fiber at \( p \). It is the principal aim of this course to explain the geometry of \( M \) at \( p \mid D(B) \) and thereby prove the above proposition.

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1.6 Further remarks

i) By global class field theory

$$\{\text{Indefinite Quaternion Algebras}/\mathbb{Q}\} \cong \{\text{Even numbers of primes}\}$$

$$B \mapsto D(B)$$

The case $D(B) = 1$ corresponds to $B = M_2(\mathbb{Q})$. All maximal orders are conjugate.

ii) We refer to Deligne-Mumford [1] or Laumon-Moret Bailly for details concerning stacks.

iii) Presentation of $\mathcal{M}$:

Fix an integer $N$. A level-$N$ structure on a pair $(A, \iota)$ is the choice of an $O_B$-linear isomorphism

$$\eta: O_B/N O_B \cong A[N].$$

Define the stack $\mathcal{M}_N/\text{Spec } \mathbb{Z}[N^{-1}]$ as

$$\mathcal{M}_N(S) = \{(A, \iota) \in \mathcal{M}(S) + \eta \text{ level-}N\text{ structure for } (A, \iota)\}.$$ 

Then $\mathcal{M}_N \to \mathcal{M}[N^{-1}]$ is a $(O_B/N O_B)^\times$-torsor. If $N \gg 0$, then $\mathcal{M}_N$ is a scheme. Thus $\mathcal{M}$ is locally (on $\text{Spec } \mathbb{Z}$) the quotient of a scheme by a finite group.

2 The degree of special cycles

2.1 Uniformization of $\mathcal{M}_\mathbb{C}$

We now want to determine the geometric generic fiber $\mathcal{M}_\mathbb{C}$. Properties of $\mathcal{M}_\mathbb{C}$ will extend to almost all fiber of $\mathcal{M}/\text{Spec } \mathbb{Z}$. We start with some preparations.

Preparations: The algebra $B$ is endowed with the canonical main involution $b \mapsto b^\tau$. It is characterized by either of the following properties

- $b \cdot b^\tau = \text{Nrd}(b)$.
- $b + b^\tau = \text{trd}(b)$.
- $(T - b)(T - b^\tau) = \text{chard}(b)(T)$. 
For all quadratic extensions $E/\mathbb{Q}$ and all embeddings $E \subset B$, $\tau|_E = \sigma_{E/\mathbb{Q}}$.

Thus $\tau \otimes \text{id}_R$ equals

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mapsto 
\begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}.
$$

Let $\delta \in \mathcal{O}_B$ be such that $\delta^\tau = -\delta$ and $\delta^2 < 0$ and define

$$b^* := \delta b \delta^{-1}.$$

Then $*$ is a positive involution, i.e. $\text{tr}(xx^*) > 0$ for all $0 \neq x \in M_2(\mathbb{R})$.

On $B$ we define the $\mathbb{Q}$-valued non-degenerate alternating form

$$(x, y) := \text{tr}(y^\tau \delta x).$$

Recall that $C = \text{End}_B(B)$ and $G = C^\times$. Obviously $G = \text{GSp}_B(\mathbb{Q}, \langle , \rangle)$, i.e. all $B$-linear automorphisms of $B$ preserve the form $( , )$ up to a constant.

**Lemma 2.1.** Let $h : \mathbb{C} \longrightarrow C \otimes \mathbb{R}$ be an $\mathcal{O}_B$-linear complex structure on $B \otimes \mathbb{R}$. Then $B \otimes \mathbb{R}/\mathcal{O}_B$ is an abelian surface (i.e. projective as complex variety). Conversely, any $(A, i) \in \mathcal{M}(\mathbb{C})$ is of this form.

Recall the following theorem about the projectivity of complex tori.

**Theorem 2.2** (Riemann). Let $T := \mathbb{C}^g/\Lambda$ be a complex torus. Then $T$ is projective if and only if there exists a non-degenerate, alternating pairing $( , ) : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ with the following properties.

- It is compatible with the complex structure, i.e. $(i \cdot, i \cdot) = ( , )$.
- The symmetric pairing $( , i \cdot)$ is positive definite.

**Proof of the lemma.** We apply this to our situation. The identity $G = C^\times$ implies that $h(i) \in G(\mathbb{R})$ is compatible with $( , )$, i.e. satisfies $( , ) = (h(i), h(i))$. One can check that $( , h(i))$ is definite so that there exists a polarization.

We now check that all points in $\mathcal{M}(\mathbb{C})$ are of this form. Let $\mathcal{O}_B \otimes \mathbb{C}^2/\Lambda$ be a complex torus with action of $\mathcal{O}_B$. Then $\mathcal{O}_B$ acts on $\Lambda$.

**Lemma 2.3.** The algebra $\mathcal{O}_B$ has class number one, i.e. every $\mathcal{O}_B$-module which is free of rank 4 as $\mathbb{Z}$-module is free of rank one over $\mathcal{O}_B$.

Let us fix an identification $\Lambda \cong \mathcal{O}_B$. This identifies $\mathbb{C}^2 \cong \Lambda \otimes_\mathbb{Z} \mathbb{R}$ with $B \otimes_\mathbb{Q} \mathbb{R}$. The complex structure on $\mathbb{C}^2$ now arises from an $h : \mathbb{C} \longrightarrow C$ and thus defines an abelian variety.

By Skolem-Noether, all homomorphisms $h : \mathbb{C} \longrightarrow C \otimes \mathbb{R}$ are conjugate. Let us fix an isomorphism $C \otimes \mathbb{R} \cong M_2(\mathbb{R})$. Then it is well-known that the conjugacy class $h$'s can be identified with $h^\pm = \mathbb{C} \setminus \mathbb{R}$. The action of $(C \otimes \mathbb{R})^\times \cong \text{GL}_2(\mathbb{R})$ on $h^\pm$ is then via Möbius transformations

**Proposition 2.4.** Let $\Gamma := \mathcal{O}_C^\times \subset G(\mathbb{R})$. Then there is an isomorphism of orbifolds

$$[\Gamma \backslash h^\pm] \longrightarrow \mathcal{M}^\text{an}_C.$$
Sketch of proof. Lemma 2.1 essentially yields a bijection \( \Gamma \backslash \mathfrak{h}^\pm \cong \mathcal{M}(\mathbb{C}) \). Now one can construct the abelian surface \( ((B \otimes_{\mathbb{Q}} \mathbb{R})/O_B,h) \) in a family over \( \mathfrak{h}^\pm \) which yields a holomorphic map
\[
\mathfrak{h}^\pm \rightarrow \mathcal{M}_C^{an}.
\]
One can check that this yields an isomorphism of orbifolds. \( \Box \)

**Corollary 2.5.** The generic fiber of \( \mathcal{M} \) is a smooth curve, geometrically connected. If \( B \) is division, then this curve is projective.

**Proof.** Connectedness follows since \( \Gamma \) contains elements with negative discriminant which interchange the two half-planes of \( \mathfrak{h}^\pm \). Projectivity since \( \Gamma \) has no cusps if \( B \not\cong M_2(\mathbb{Q}) \). \( \Box \)

**Remark 2.6.** The moduli problem with non-trivial level-structure (over \( \mathbb{C} \)) is not necessarily connected. It is Lemma 2.3 which fails when incorporating level structure.

### 2.2 Uniformization of \( Z(t)_C \)

Let \((A,\iota) \in \mathcal{M}(\mathbb{C})\). The choice of an isomorphism \( A = (B \otimes \mathbb{R})/O_B \) yields a point \( h \in \mathfrak{h}^\pm \) and identifies
\[
\text{End}(A,\iota) = \text{Cent}_h(O_C).
\]
Define \( L := \{ x \in O_C \mid \text{tr}(x) = x^\tau + x = 0 \} \) and \( L(t) := \{ x \in L \mid Q(x) = -x^2 = x^\tau x = t \} \). For \( x \in L(t) \), let
\[
D_x := \{ h \mid xh = hx \}.
\]
Then \( |D_x| = 2 \). Namely let \( k := \mathbb{Q}(\sqrt{-t}) \). Then \( x \) defines an embedding
\[
i_x : k \hookrightarrow C
\]
such that \( \mathbb{Z}[\sqrt{-t}] \subset i_x^{-1}(O_C) \). But the centralizer of \( k \subset C \) is \( k \) itself. So \( D_x = \text{Hom}_C(k,k \otimes \mathbb{R}) \). We define
\[
D_{Z(t)} := \bigcup_{x \in L(t)} D_x \subset \mathfrak{h}^\pm.
\]
The set \( D_{Z(t)} \) is stable under the action of \( \Gamma \). More precisely, \( \gamma \in \Gamma \) induces an isomorphism \( \gamma : D_x \rightarrow D_{xy^{-1}} \). There is a 2 : 1-surjection
\[
D_{Z(t)} \twoheadrightarrow L(t)
\]
which is \( \Gamma \)-equivariant. Here \( \Gamma \) acts by conjugation on \( L(t) \).

**Proposition 2.7.** There is an isomorphism of orbifolds
\[
[\Gamma \backslash D_{Z(t)}] \xrightarrow{\cong} Z(t)_C.
\]
Of course, this isomorphism is compatible with the uniformization with of \( \mathcal{M}_C \).
2.3 The degree of $Z(t)_C$

The reference is Part III of [13].

To state the result, we need some notation. Let $k = \mathbb{Q}(\sqrt{-t})$ have discriminant $d$. The order $\mathbb{Z}[\sqrt{-t}]$ has discriminant $4t = n^2d$ where $n$ denotes its conductor. In general, we denote by $O_{c,d} \subset k$ the order of conductor $c$. We let $h(c^2d)$ be its class number and $w(c^2d) := |O_{c,d}^2|$ be the number of roots of unity in $O_{c,d}$.

We define

$$H_0(t, D) := \sum_{c|n,(c,D) = 1} \frac{h(c^2d)}{w(c^2d)}.$$ 

This is a variant of the Hurwitz class number, counting certain isomorphism classes of quadratic lattices, weighted with the number of their automorphisms.

We also define

$$\delta(d, D) := \prod_{l|D}(1 - \chi_d(l)).$$ 

where $\chi_d$ denotes the quadratic character associated to $k$, i.e.

$$\chi_d(l) = \begin{cases} 
1 & \text{if } l \text{ split in } k \\
-1 & \text{if } l \text{ inert in } k \\
0 & \text{if } l \text{ ramified in } k.
\end{cases}$$

Then $\delta(d, D) \neq 0$ if and only if there exists an embedding $k \hookrightarrow B$.

**Proposition 2.8.** Let $k := \mathbb{Q}(\sqrt{-t})$.

There is an equality

$$\deg Z(t)_C = 2\delta(d, D)H_0(t, D).$$

In particular, $Z(t)_C = \emptyset$ if $k$ does not embed into $B$.

**Proof.** It is clear that $k$ embeds into $B$ if and only if $Z(t)_C \neq 0$.

By the explicit description of $Z(t)_C$,

$$\deg Z(t)_C = 2 \sum_{x \in (L(t))/\Gamma} \frac{1}{|\Gamma_x|}.$$ 

Here the $2$ is due to the fact that $D_{Z(t)}$ is $2 : 1$ over $L(t)$. As mentioned above, each $x$ defines $i_x : k \rightarrow C$ such that $\mathbb{Z}[\sqrt{-t}] \subset i_x^{-1}(O_C)$. Let $e$ be the conductor of $i_x^{-1}(O_C) = O_{c,d}$. Then $e | n$ and $(c, D) = 1$ since the order $i_x^{-1}(O_C)$ is maximal at all places $p | D$. The conductor is preserved under the action of $\Gamma$ on $x$.

Let $\text{Opt}(O_{c,d}, O_C) := \{i : k \rightarrow C \mid i^{-1}(O_B) = O_{c,d}\}/\Gamma$ be the set of optimal embeddings.

**Lemma 2.9 (Eichler, 9.6).** There is an equality

$$|\text{Opt}(O_{c,d}, O_C)| = \delta(d, D)h(c^2d).$$
Thus we see
\[
\sum_{x \in \Gamma \setminus L(t)} |\Gamma_x|^{-1} = \sum_{c|n, (c,D)=1} |\text{Opt}(O_{c^2d}, O_C)| \cdot |O_{c^2d}|^{-1} = \delta(d,D) \sum_{c|n, (c,D)=1} h(c^2d)/w(c^2d) = \delta(d,D)H_0(t,D).
\]

\[\square\]

2.4 Comparison with an Eisenstein series

The point is, that modularity of $\phi_{\text{deg}}$ is not proved by checking the transformation property. Instead, one compares the degrees which were computed above with the Fourier coefficients of an explicit Eisenstein series

$$E(\tau, s; D), \quad \tau \in \mathfrak{h}, s \in \mathbb{R}_{>0}$$

This series is defined in a canonical way through an adelic formalism in Chapter 6 of [13]. For fixed $s$, this function transforms like a modular form of weight $1 + s$. A direct computation (see Chapter 8 of [13]) shows that

$$E(\tau, \frac{1}{2}; D) = -\frac{1}{12} \prod_{p|D} (p - 1) + \sum_{m>0} 2\delta(d; D)H_0(m; D)q^m.$$

This implies the modularity of $\phi_{\text{deg}}$.

3 Properness of $M$

3.1 Valuative Criteria for stacks

Recall the following definitions, see [1].

**Definition 3.1.** Let $S$ be noetherian scheme and $f : X \to S$ Deligne-Mumford stack of finite type.

i) $f$ is separated if the diagonal $X \to X \times_S X$ is proper.

ii) $f$ is proper if it is of finite type, separated and universally closed.

**Remark 3.2.** Universally closed refers to the underlying Zariski spaces $|X'| \to |S'|$ for each scheme $S'/S$.

Now by definition, the diagonal is representable so the valuative criterion for properness yields the valuative criterion for separatedness.

**Proposition 3.3** (Valuative Criterion for Separatedness). Let $S$ be a noetherian scheme, $f : X \to S$ of finite type with separated quasi-compact diagonal $X \to X \times_S X$. Then $f$ is separated if and only if:

For all DVR $R$ over $S$ and any two morphisms $g_1, g_2 : \text{Spec } R \to X$ over $S$, any isomorphism between the restrictions of $g_1$ and $g_2$ to the generic point lifts to $R$.

Similarly, we have a valuative criterion of properness. (You can take this as definition if you wish.)
Proposition 3.4 (Valuative Criterion of Properness). \( S \) noetherian scheme, \( f : X \to S \) of finite type and separated. Then \( f \) is proper if and only if:

For all DVR \( R \) over \( S \) with field of fractions \( K \) and all \( g : \text{Spec} \ R \to X \) over \( S \), there exists a finite extension \( K'/K \) such that \( g \) lifts to \( R' \), the integral closure of \( R \) in \( K' \).

In both propositions, we can restrict to complete \( R \) with algebraically closed residue field.

3.2 Néron-models and the semi-stable reduction theorem

The reference is [3][Chap. 7].

We would like to apply these criteria to moduli of abelian varieties. Let \( R \) be a DVR with field of fractions \( K \) and residue field \( k \). Let \( A_K \) be an abelian variety over \( K \).

Definition 3.5. A Néron model of \( A_K \) over \( R \) is a smooth separated finite type scheme \( X' \to \text{Spec} \ R \) such that \( A_K \cong X \otimes_R K \) with the following universal property. For all smooth schemes \( T \to \text{Spec} \ R \), every morphism \( T \otimes_R K \to A_K \) extends uniquely to \( T \to X \).

Remark 3.6. i) A Néron model is unique up to unique isomorphism. 

ii) By the universal property, the group structure of \( A_K \) lifts to the Néron model \( X \).

Theorem 3.7 (See Chapter 1 of BLR). i) Any abelian variety \( A_K/K \) admits a Néron model \( A/R \).

ii) An abelian scheme \( A/R \) is a Néron model of its generic fibre.

Note that the Néron model of an abelian variety is not generally an abelian scheme. Indeed, the definition does not include properness.

Example 3.8. Let \( E_K/K \) be an elliptic curve with Néron model \( E/R \). Let \( E^0_k \) be the connected component of 0 of the special fiber of \( E \). There are three (mutually exclusive) possibilities.

i) \( E \) is an elliptic curve over \( R \) (good reduction).

ii) \( E^0 \) is a torus (multiplicative/semi-stable reduction).

iii) \( E^0_k \cong G_a \) (bad/unstable reduction).

Note that formation of the Néron model does not necessarily commute with extending the base field \( K'/K \).

Theorem 3.9 (Grothendieck, Semi-stable Reduction theorem). Let \( R \) be a DVR with field of fractions \( K \) and let \( A_K/K \) be an abelian variety. Then \( A_K \) has potential semistable reduction. This means, there exists a finite extension \( K'/K \) such that the Néron model of \( A_K \otimes_K K' \) over \( R' \) has a special fiber, which is an extension of an abelian variety by a torus.

Example 3.10. Let \( E_K/K \) be an elliptic curve. Then after a suitable base change \( K'/K \), the elliptic curve \( E_K \otimes_K K' \) will fall into case i) or ii).

3.3 Application to \( M \)

If \( B \cong M_2(\mathbb{Q}) \), then \( M \) is the modular curve. We know that it is not proper over \( \text{Spec} \ \mathbb{Z} \). It has to be compactified at the cusp. If \( B \not\cong M_2(\mathbb{Q}) \), we have the following result.
Proposition 3.11. Let $B \neq M_2(\mathbb{Q})$. Then the Shimura curve $\mathcal{M}/\text{Spec } \mathbb{Z}$ associated to $B$ is proper.

Proof. We verify the valuative criteria above.

Separatedness: The conditions of the criterion are satisfied. The separatedness follows from the Néron property for abelian schemes.

Properness: Let $(A_K, \iota_K) \in \mathcal{M}(K)$ and let $K'$ be a finite extension from the semi-stable reduction theorem. Let $A$ be the Néron model over $R'$. The action of $O_B$ lifts to $A$ by the universal property. Let

$$0 \rightarrow T \rightarrow A_0^0 \rightarrow B \rightarrow 0$$

be exact with $B$ an abelian variety over the residue field of $R$. We need to show that $\dim T = 0$. It is a basic fact that $\text{Hom}(T, B) = 0$. ($T$ cannot surject onto an elliptic curve, which would be the image of a non-trivial homomorphism.) So the action of $O_B$ on $A_0^0$ induces an action of $O_B$ on $T$. But $O_B$ cannot map to $M_2(\mathbb{Z})$. (Here the assumption $B \neq M_2(\mathbb{Z})$ enters.) Thus $\dim T = 0$ and $A_K \otimes_R K'$ has good reduction. So the point $(A_K, \iota_K)$ lifts to $R'$. By the valuative criterion, $\mathcal{M}/\text{Spec } \mathbb{Z}$ is proper. \qed

4 Structure of $\mathcal{M}[D(B)^{-1}]$ and $\mathcal{Z}(t)[D(B)^{-1}]$

We have seen that the generic fibre $\mathcal{M}_{\mathbb{C}}$ is smooth. For abstract reasons, this extends to almost all fibers. We will now show that $\mathcal{M}$ is smooth away from the discriminant.

Proposition 4.1. The morphism $\mathcal{M}[D(B)^{-1}] \rightarrow \text{Spec } \mathbb{Z}[D(B)^{-1}]$ is smooth of relative dimension 1.

Corollary 4.2. $\mathcal{M}[D(B)^{-1}]$ is a regular 2-dimensional stack. It is flat with geometrically connected fibers over $\mathbb{Z}[D(B)^{-1}]$.

Note that we cannot prove this fiber by fiber since we do not know flatness of $\mathcal{M}/\text{Spec } \mathbb{Z}$. But we can check the smoothness of the fibre over $p \nmid D(B)$ at each closed point of $\mathcal{M} \otimes \mathbb{F}_p$. So let $\mathbb{F} := \mathbb{F}_p$ and $(A, \iota) \in \mathcal{M}(\mathbb{F})$. Let $W := W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$. It equals the strict completion of $\mathbb{Z}_p$. Let $\hat{\mathcal{O}}_{\mathcal{M},(A,\iota)}$ be the strict local ring in $(A, \iota)$. Then we need to show the induced morphism on formal schemes

$$\text{Spf } \hat{\mathcal{O}}_{\mathcal{M},(A,\iota)} \rightarrow \text{Spf } W$$

is formally smooth. The source is of course the formal deformation space of $(A, \iota)$.

The crucial point is that the deformation space of $(A, \iota)$ only depends on the $p$-divisible group of $A$. We will explain this now.

4.1 $p$-divisible groups and the theorem of Serre-Tate

A reference is [7].

Definition 4.3. Let $S$ be a scheme. A $p$-divisible group of height $h$ over $S$ is an inductive system $(X_h)_{h \in \mathbb{N}}$ of finite flat commutative group schemes $X_h/S$ such that
i) $X_n$ has order $p^n$.
ii) For all $n$, the following sequence is exact:

$$0 \rightarrow X_n \rightarrow X_{n+1} \rightarrow \mu_p^n \rightarrow X_{n+1}.$$ 

If $X = (X_n)/S$ is a $p$-divisible group (and $S$ connected), we define the dimension $\dim X := \mathrm{rk}_{O_S}(\operatorname{Lie}(X_1))$.

**Example 4.4.** i) $\mathbb{Q}_p/\mathbb{Z}_p := \lim \leftarrow p^{-n}\mathbb{Z}/\mathbb{Z}$ is a $p$-divisible group of height 1 and dimension 0. Here $X_n$ is the constant group scheme $X_n := p^{-n}\mathbb{Z}/\mathbb{Z}$. The transition maps $X_n \rightarrow X_{n+1}$ are induced from the inclusions $p^{-n}\mathbb{Z} \hookrightarrow p^{-n-1}\mathbb{Z}$.

ii) $\mu_{p^n} := \lim \leftarrow \mu_{p^n}$ is of height 1 and dimension 1. Here $X_n$ is the $p^n$-torsion of $G_m$.

iii) Let $A$ be an abelian variety. Then $A(p) := \lim \rightarrow A[p^n]$ is a $p$-divisible group of height 2 $\dim A$ and dimension $\dim A$.

For two $p$-divisible groups $X = (X_n)_n$ and $Y = (Y_n)_n$, we have

$$\operatorname{Hom}(X, Y) = \lim_{\leftarrow} \operatorname{Hom}(X_n, Y_n).$$

Note that $\mathbb{Z}$ and hence $\mathbb{Z}_p$ acts naturally on each $p$-divisible group. So $\operatorname{Hom}(X, Y)$ is a $\mathbb{Z}_p$-module. For example, if $S$ is connected and $p$ locally nilpotent in $O_S$, then $\operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^n}) = \{0\}$ and $\operatorname{End}(\mu_{p^n}) = \operatorname{End}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p$.

If $A, B$ are abelian varieties, then

$$\operatorname{Hom}(A, B) \leftarrow \operatorname{Hom}(A, B) \otimes \mathbb{Z}_p \leftarrow \operatorname{Hom}(A(p), B(p)).$$

**Theorem 4.5** (Serre-Tate). Let $A/S$ be an abelian scheme with $p$ locally nilpotent in $O_S$. Let $S \rightarrow S'$ be a nilpotent thickening.

i) There is a bijection

$$\{\text{Deformations of } A \text{ to } S'\} = \{\text{Deformations of } A(p) \text{ to } S'\}$$

$$A' \mapsto A'(p)$$

ii) Let $A'$ and $B'$ be two abelian schemes over $S'$ and let $f : A' \times_{S'} S \rightarrow B' \times_{S'} S$ be a homomorphism. Then $f$ lifts to a homomorphism $A' \rightarrow B'$ if and only if $f(p)$ lifts to $A'(p) \rightarrow B'(p)$.

**Remark 4.6.** Outside of characteristic $p$, the $p$-divisible group $A(p)$ encodes the same information as the $p$-adic Tate-module $T_p(A)$.

**Example 4.7.** Let $\text{Ell}/\text{Spec } \mathbb{Z}$ be the moduli stack of elliptic curves. It is smooth of dimension 1 over $\mathbb{Z}$. This follows immediately from the infinitesimal criterion since there is no obstruction to lifting the coefficients of a Weierstrass equation.

Now let $X := E(p)/\mathbb{F}$ be a $p$-divisible group which appears as the $p$-torsion of an elliptic curve. Then by the Theorem of Serre-Tate, its deformation space over $\text{Spf } W(\mathbb{F})$ is formally smooth of dimension 1, i.e. isomorphic to $\text{Spf } W(\mathbb{F})[[t]]$.

**Definition 4.8.** Let $X$ and $Y$ be two $p$-divisible groups over a (qc) scheme $S$. A quasi-homomorphism is an element of

$$\operatorname{Hom}^0(X, Y) := \operatorname{Hom}(X, Y) \otimes \mathbb{Q}.$$ 

A quasi-isogeny is an invertible quasi-homomorphism.
Theorem 4.9 (Dieudonné-Manin). Let $k$ be an algebraically closed field in characteristic $p$ and let $(\text{pdiv}^0/k)$ be the category of $p$-divisible groups with quasi-homomorphism. Then $(\text{pdiv}^0/k)$ is semi-simple and there is a bijection

$\mathbb{Q} \cap [0, 1] \cong \Sigma(\text{pdiv}^0/k)$

$\lambda \mapsto X_\lambda.$

If $0 \leq r \leq s$ such that $(r, s) = 1$, then $X_{r/s}$ has height $s$ and dimension (of the tangent space) $r$.

Definition 4.10. Let $X/k$ be a $p$-divisible group. We decompose it into simple factors in the isogeny category

$X \cong X_0 \times \ldots \times X_n.$

Let $X_i$ correspond to the rational number $\lambda_i$ under the Dieudonné-Manin classification. Let us assume that $\lambda_i \leq \lambda_{i+1}$. Then the sequence

$(\lambda_0, \lambda_1, \ldots, \lambda_n)$

is called the slope sequence or slopes of $X$.

Example 4.11. The group $\mathbb{Q}_p/\mathbb{Z}_p$ has height 1 and dimension 0, so it has slope $0/1$. The group $\mu_{p^n}$ has slope $1/1$.

Example 4.12. Let $E/k$ be an elliptic curve. There are two possibilities for the slopes of $X := E(p)$, which has height 2 and dimension 1. These are $(0/1, 1/1)$ and $(1/2, 1/2)$. The first case corresponds to $E$ being ordinary, the second to $E$ being supersingular. Note that this yields a way to define $X_{1/2}$. Also note, that the two groups of height 2 and dimension 1 are even unique up to isomorphism, not only up to isogeny.

To any $p$-divisible group $X = (X_n)/S$ we associate its dual $X^\vee := \lim \to X_n^\vee$. Here $X_n^\vee$ is the dual in the category of finite commutative group schemes. The transition maps of the inductive systems are the duals of multiplication by $p$

$p \cdot \text{id}_{X_{n+1}} : X_{n+1} \to X_n.$

If $X/\text{Spec } k$ is simple and $X \leftrightarrow r/s$ under the Dieudonné-Manin theorem, then $X^\vee \leftrightarrow (s-r)/s$. Furthermore,

$A(p)^\vee \cong A^\vee(p)$

are canonically isomorphic, where $A^\vee$ denotes the dual abelian variety.

Corollary 4.13. Let $A/k$ be an abelian variety over an algebraically closed field of characteristic $p$. Let $\lambda_0 \leq \ldots \leq \lambda_n$ be the slopes of $A(p)$ (with multiplicities). Then

$\lambda_i = 1 - \lambda_{n-i}.$

Proof. Let $A^\vee/k$ be the dual abelian variety. Since $A$ is projective, there exists an isogeny $A \to A^\vee$. Thus their $p$-divisible groups are isogeneous,

$A(p) \sim A^\vee(p) \cong A(p)^\vee.$

Now we compare the slopes of $A(p)$ and $A^\vee(p)$.

Corollary 4.14. Let $A/k$ be an abelian surface and set $X := A(p)$. Then the only possibilities for the slopes of $A(p)$ are

$(0, 0, 1, 1), (0, 1/2, 1) \text{ and } (1/2, 1/2).$
4.2 Application to $M$

**Proof of the smoothness.** Let $(A, i) \in \mathcal{M}(\mathbb{F})$ where $\mathbb{F} = \mathbb{F}_p^{alg}$. We want to compute the deformation space of $(A, i)$. This is the same as the deformation space of $(A(p), i(p))$, where $i(p)$ is the induced action of $O_B \otimes \mathbb{Z}_p$ on $A(p)$.

Now $p \nmid D(B)$ and hence $O_B \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$. Using the idempotents of $M_2(\mathbb{Z}_p)$, we can write $A(p) = Y^2$ for some $p$-divisible group of height 2. Then deformations of $(A, i)$ are in bijection with deformations of $Y$ (without any action!).

We have seen above that any group of height 2 and dimension 1 embeds into an elliptic curve. Thus $Def(Y)$ is formally smooth of relative dimension 1 over $W(\mathbb{F})$.

**Remark 4.15.** In general, if $X/\mathbb{F}$ is a $p$-divisible group of height $h$ and dimension $d$, then its deformation space is formally smooth of relative dimension $h(h - d)$ over $W(\mathbb{F})$. This follows from Theorem 4.18 below.

4.3 Deformation theory of $p$-divisible groups

We collect some important results. In this subsection $p$ is some prime and $S$ is a $\text{(qc)}$ scheme with $p$ locally nilpotent in $O_S$.

**Proposition 4.16 (Unramifiedness of $\text{Hom}$).** Let $X$ and $Y$ be $p$-divisible groups over $S$ and let $f : X \rightarrow Y$ be a quasi-homomorphism. Then there exists a closed subscheme $Z(f) \subset S$ such that $u : T \rightarrow S$ factors over $Z(f)$ if and only if $u^*f$ is a homomorphism $u^*X \rightarrow u^*Y$.

**Proposition 4.17 (Drinfeld, Rigidity).** Let $S \hookrightarrow S'$ be a nilpotent thickening. Let $X$ and $Y$ be two $p$-divisible groups over $S'$. Then the reduction of homomorphisms defines a bijection of quasi-homomorphisms

$$\text{Hom}^0(X, Y) \xrightarrow{\sim} \text{Hom}^0(X \times_{S'} S, Y \times_{S'} S).$$

Injectivity follows from the unramifiedness of $\text{Hom}$. Surjectivity can be formulated as follows. Let $f : X \times_{S'} S \rightarrow Y \times_{S'} S$ be a homomorphism. Then there exists (locally on $S$) an integer $N$ such that $p^Nf$ extends to $S'$.

**Theorem 4.18 (Grothendieck, Crystalline Deformation Theorem).** Let $S \hookrightarrow S'$ be a square-zero thickening, i.e., $S = V(I)$ for some ideal $I \subset O_S$ with $I^2 = 0$. (More generally, $S \subset S'$ is a PD-thickening.) Then there exists a functor

$$\mathbb{D}_S : (p\text{div}/S) \rightarrow (O_S\text{-modules + filtration})$$

$$X \mapsto (\mathbb{D}_S(X), F \subset \mathbb{D}_S(X) \otimes_{O_S} O_S)$$

with the following properties. The $O_S$-module $\mathbb{D}_S(X)$ is locally free of rank equal to the height of $X$. And $F \subset \mathbb{D}_S(X) \otimes O_S$ is locally a direct summand, locally free of rank $\dim(X^\vee) = (\text{ht}(X) - \dim(X))$. Furthermore, there is a bijection

$$\{\text{Lifts } F' \text{ of } F \text{ to locally direct summands of } \mathbb{D}_S(X)\} \cong \{\text{Deformations of } X \text{ to } S'\}.$$  

If $X'/S'$ is the deformation corresponding to $F'$ and if $f \in \text{End}(X)$, then $f$ lifts to $X'$ if and only if $\mathbb{D}(f)F' \subset F'$.

Terminology: $\mathbb{D}_S(X)$ is called the crystal of $X$ evaluated at $S'$. The submodule $F$ is called the Hodge filtration.
4.4 Application to $\mathcal{Z}(t)$

**Proposition 4.19.** The $0$-cycle $\mathcal{Z}(t)\subset$ is nonempty if and only if the imaginary quadratic field $\mathbb{Q}(\sqrt{t})$ embeds into $B$. In this case, the stack $\mathcal{Z}(t)$ is flat over $\text{Spec} \mathbb{Z}_{[D(B)^{-1}]}$ and represents a relative divisor.

*Proof.* The first assertion was explained in the first chapter. The second assertion can be proved after completion.

Let $p \nmid D(B)$, $F = \overline{F}_p$ and $W = W(F)$ as usual. Let $(A, \iota, x) \in \mathcal{Z}(t)(F)$. It is enough to show that $\text{Def}(A, \iota, x) \to \text{Def}(A, \iota)$ is a relative divisor over $\text{Spf} W$.

**Step 1: Reduction to $p$-divisible groups.** Let $X := A(p)$ be the $p$-divisible group of $A$. It has height 4 and dimension 2. It is endowed with the induced action $\iota : O_B \to \text{End}(X)$ and the induced endomorphism $x \in \text{End}_{O_B}(X)$. The action $O_B \odot X$ extends to an action of $O_B \otimes \mathbb{Z}_p$. By Serre-Tate, $\text{Def}(A, \iota, x) = \text{Def}(X, \iota, x)$ and $\text{Def}(A, \iota) = \text{Def}(X, \iota)$.

Now $p \nmid D(B)$, so $O_B \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$. Choosing two complementary idempotents yields a decomposition $(X, \iota, x) = (Y^2, \text{can}, x)$. Here $Y$ is some $p$-divisible group of height 2 and dimension 1 and $\text{can}$ is the canonical action of $M_2(\mathbb{Z}_p)$ on $Y^2$. The endomorphism $x$ is $O_B$-linear and so is diagonal, $x = (y, y)$. Then $\text{Def}(X, \iota, x) \subseteq \text{Def}(X, \iota)$ equals $\text{Def}(Y, y) \subseteq \text{Def}(Y)$.

**Claim:** Let $Y$ be a $p$-divisible group of height 2 and dimension 1. Let $y \in \text{End}(Y) \setminus \mathbb{Z}_p$. Then $\text{Def}(Y, y) \subset \text{Def}(Y)$ is a relative divisor (over $\text{Spf} W(F)$).

**Step 2:** $\text{Def}(Y, y) \subset \text{Def}(Y)$ is a closed formal subscheme. Proposition 4.16 yields a unique way to extend $y$ to a quasi-endomorphism on $\text{Def}(Y)$. Then Proposition 4.16 yields the result.

**Step 3:** $\text{Def}(Y, y) \subset \text{Def}(Y)$ is generated by one equation. Let $S'/W$ be an artinian and consider $Y' \in \text{Def}(Y)(S')$ a deformation of $Y$. Let $S \to S'$ be a square-zero closed subscheme such that $y$ lifts to $Y' \times_{S'} S$. Let

$$0 \to F' \to D_S(Y') \to Q' \to 0$$

be the Hodge filtration of $Y'$, as in Theorem 4.18. Let $\mathbb{D}(y)$ be the endomorphism of the Dieudonné crystal $D(Y')$ induced by $y$. Then $\mathbb{D}(y)F \subset F$, i.e. $y$ preserves the Hodge filtration of $Y' \times_{S'} S$.

By 4.18, $y$ lifts to $Y$ if and only if $\mathbb{D}(y)F' \subset F'$. Equivalently, if the composition

$$F' \to D_S(Y') \xrightarrow{\mathbb{D}(y)} D_S(Y') \to Q'$$

is zero. But this is a homomorphism of “line bundles” (on the zero-dimensional scheme $S'$), so its vanishing locus is obviously described by one equation. Note that we did not use $y \notin \mathbb{Z}_p$. If this is the case, then the defining equation is just 0.

**Step 4:** $\text{Def}(Y, y) \subset \text{Def}(Y)$ is a relative divisor.

The deformation space $\text{Def}(Y) \cong \text{Spf} W(F[[v]])$ is regular, so it is enough to show that $\text{Def}(Y, y) \neq \text{Def}(Y)$ to prove that $\text{Def}(Y, y)$ is a divisor. This divisor is regular if and only if $V(y) = \text{Spf} F[[v]] \notin \text{Def}(Y, y)$. The claim now follows from the results in [2, Chap. 8 and 9].
Note that if we are only interested in a special endomorphism $y$ induced from a point $(A, \iota, x) \in \mathcal{Z}(t)(\mathbb{F})$, then it is enough to understand the supersingular case, i.e. [2, Chap. 8]. Namely, $\text{Def}(Y, y)$ is then the formal deformation space of $(A, \iota, x)$ and it is enough to show that

$$\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p \not\subseteq \text{Image}(\mathcal{Z}(t) \rightarrow \mathcal{M}).$$

But this can be checked at a supersingular point in $\mathcal{M}(|\mathbb{F}|)$.

5 Structure of $\mathcal{M}$ at $p \mid D(B)$ (p-adic uniformization)

The reference for this section is [6]. Everything can be found there.

The complex fiber $\mathcal{M}_p$ has a uniformization by the upper half-plane. Similarly if $p \mid D(B)$, then the p-adic completion $\mathcal{M}_p$ of $\mathcal{M}$ along $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$ has a uniformization by the Drinfeld upper half-plane $\Omega$.

We will now explain this p-adic uniformization. To do this, we also have to define $\Omega$ and to explain the concept of a Rapoport-Zink space.

5.1 The theorem of Honda and Tate

Recall that an isogeny $A \rightarrow B$ of abelian varieties is a surjective homomorphism with finite kernel. A quasi-isogeny is an invertible element in $\text{Hom}^0(A, B) := \text{Hom}(A, B) \otimes \mathbb{Q}$. Recall that the category of abelian varieties up to quasi-isogeny over a fixed field is semi-simple.

Fix $q$ and denote by $\sum AV^0(F_q)$ the isomorphism classes of simple objects of the category of abelian varieties up to isogeny. Here, simple and isogenies are as abelian varieties over $F_q$.

Let

$$W(q) := \{ \alpha \in \overline{\mathbb{Q}} \mid \forall \rho : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}, \ |\rho(\alpha)| = q^{1/2} \}$$

be the set of Weil-$q$-numbers of weight 1. The absolute Galois group $G_{\mathbb{Q}}$ acts on it.

If $A \in \sum AV^0(F_q)$, then $\text{End}^0(A)$ is a division algebra over $\mathbb{Q}$ and the $q$-Frobenius $\pi_A$ generates a field extension $\mathbb{Q}(\pi_A)/\mathbb{Q}$. The Riemann hypothesis for varieties over $F_q$ states that $\pi_A$ is a Weil-$q$-number, $\pi_A \in G_{\mathbb{Q}} \backslash W(q)$.

**Theorem 5.1** (Honda-Tate). The map $A \mapsto \pi_A$ induces a bijection between $\sum AV^0(F_q)$ and $G_{\mathbb{Q}} \backslash W(q)$. Furthermore if $A$ corresponds to $\pi_A$, then $D_A := \text{End}^0(A)$ is a division algebra with center $\mathbb{Q}(\pi_A)$. One knows that

$$2 \dim A = [\mathbb{Q}(\pi_A) : \mathbb{Q}][D_A : \mathbb{Q}(\pi_A)]^{1/2}.$$ 

This division algebra is unramified outside $p$.

**Example 5.2.** Let $E/\mathbb{F}_q$ be a supersingular elliptic curve. Then $\pi_E$ is of degree $q$ and satisfies $\pi_E^q = q$. (A purely inseparable morphism between curves factors over the Frobenius.) It follows that $\pi_E = \sqrt{q}$.

**Lemma 5.3.** Let $p \mid D(B)$ be a prime and set $\mathbb{F} := \mathbb{F}_{p^{\text{alg}}}$. Then the points in $\mathcal{M}(\mathbb{F})$ form a single isogeny class (wrt. $O_B$-linear isogeny.)
Proof. We first show that for every \((A, \iota) \in \mathcal{M}(F)\), \(A\) is isogeneous to a product of supersingular elliptic curves.

First note that \(A(p)\) has an action of \(O_B \otimes \mathbb{Z}_p\) which is a maximal order in a quaternion algebra over \(\mathbb{Q}_p\). The \(p\)-divisible group \(A(p)\) is of dimension 2 and height 4. We decompose it into isotypical components in the isogeny category. \(B \otimes \mathbb{Q}_p\) acts on each isotypic component.

The slopes 0 and 1 cannot occur since \(B \otimes \mathbb{Q}_p\) can act on no smaller power than \((\mathbb{Q}_p/\mathbb{Z}_p)^4\) or \(\mu_p^4\). So by Corollary 4.14, the slopes are \((1/2, 1/2)\) and \(A\) is supersingular. Now assume that \(A\) is defined over \(\mathbb{F}_q\). It follows from the structure of \(A(p)\) that \(\pi_A^2/q\) is integral such that \(|\rho(\pi_A^2/q)| = 1\) for all embeddings \(\rho : \mathbb{Q}(\pi_A) \to \mathbb{C}\). It is thus a root of unity. By extending scalars, we can assume \(\pi_A = \sqrt{q} \in \mathbb{Q}\).

But this Weil number corresponds to a supersingular elliptic curve under the Honda-Tate bijection. It follows that \(A\) cannot be simple. It is thus isogeneous to a product of two supersingular elliptic curves, \(A \sim E \times E\).

Via this isogeny, we transport the \(O_B\)-action to \(E \times E\). This is only an action by quasi-isogenies! But up to isogeny, there is a unique action \(B \to \End^0(E \times E)\). Namely recall that \(\End^0(E \times E) \cong D(0)/\mathbb{Q}\), the quaternion algebra with invariant 1/2 precisely at \(p\) and \(\infty\). By Skolem-Noether, all embeddings \(B \to M_2(D(0))\) are conjugate. This finishes the proof. \(\square\)

From here on, we fix a base point \((A_0, \iota_0) \in \mathcal{M}(F)\). We have seen that all points in \(\mathcal{M}(F)\) are isogeneous to \((A_0, \iota_0)\). So we can overparametrize \(\mathcal{M}(F)\) by parametrizing all \(O_B\)-linear quasi-isogenies \((A_0, \iota_0) \to (A, \iota)\). We now motivate the characteristic \(p\) theory with the theory over \(\mathbb{C}\).

### 5.2 Isogenies over \(\mathbb{C}\)

Let \(\hat{\mathbb{Z}} := \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p\) be the integral adeles and \(A_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}\). There is an exact sequence

\[
0 \to \hat{\mathbb{Z}} \to A_f \to A_f/\hat{\mathbb{Z}} \to 0.
\]

Let \(A/\mathbb{C}\) be an abelian variety of dimension \(d\). Define the Tate module \(T(A) := \lim A[n]\) and the rational Tate module \(\mathcal{V}(A) := T(A) \otimes_\mathbb{Z} A_f\). Tensoring the above sequence with \(T(A)\) yields

\[
0 \to T(A) \to \mathcal{V}(A) \to \lim A[n] \to 0.
\]

Giving an isogeny \(f : A \to B\) is the same as giving its kernel, a finite flat group scheme \(K \subset A\). If \(f\) has degree \(n\), then \(K \subset A[n]\). Then \(\alpha^{-1}(T(B))\) is a superlattice \(T(A) \subset \Lambda \subset \mathcal{V}(A)\). This yields a bijection

\[
\{\text{Quasi-isogenies } f : A \to B\} = \{\text{lattices } \Lambda \subset \mathcal{V}(A)\} = \GL(\mathcal{V}(A))/GL(T(A)).
\]

Now consider two quasi-isogenies \(f : A \to B\) and \(f' : A \to B'\) corresponding to lattices \(\Lambda, \Lambda' \subset \mathcal{V}(A)\). Then \(B \cong B'\) if and only if there exists a quasi-isogeny \(g : B \to B'\) such that \(gT(B) = T(B')\) in \(\mathcal{V}(B')\). If \(\varphi : B \cong B'\), then \(\varphi := (f')^{-1}\varphi f \in I(A)\), the group of quasi-isogenies of \(A\), and \(\psi\Lambda = \Lambda'\). Choosing a \(\hat{\mathbb{Z}}\)-basis of \(T(A)\), we arrive at:

**Remark 5.4** (Adelic description of an isogeny class over \(\mathbb{C}\)). There is a bijection

\[
\{\text{Abelian varieties } B, \text{isogeneous to } A\} = I(A)\GL_{2d}(A_f)/\GL_{2d}(\hat{\mathbb{Z}}).
\]
5.3 Isogenies in characteristic $p$ and Rapoport-Zink spaces

The reference is [16].

Now if $A$ is in char $p$, the adelic description only works for prime-to-$p$ isogenies. But at $p$, we have to replace the Tate-module by the $p$-divisible group $A(p)$. Obviously, there is a bijection

$$\{\text{finite flat } p\text{-grps } K \subset A\} \cong \{\text{subgrps } K \subset A(p)\}.$$ 

Let $\hat{\mathbb{Z}}^p$, $A_f^p$, $T^p(A)$ and $\mathbb{V}^p(A)$ be the corresponding rings/modules away from $p$. Then

$$\{\text{quasi-isogenies } A \to B\} = \{\hat{\mathbb{Z}}^p - \text{lattices } \Lambda \subset \mathbb{V}^p(A) + \text{a quasi-isogeny } A(p) \to B(p)\}.\quad (5.1)$$

This motivates the next definition. Let $W := W(F)$ be the ring of Witt vectors for $F$. Denote by $\text{Nilp}$ the category of schemes over $\text{Spf} W$, i.e., the category of schemes $S/\text{Spec} W$ with $p$ locally nilpotent in $\mathcal{O}_S$. For $S \in \text{Nilp}$, we write $\overline{S} := S \otimes W F$ for the special fiber.

**Definition 5.5.** Let $X \to \text{Spec} F$ be $p$-divisible group. On $\text{Nilp}$, we define the functor

$$N_X : \text{Nilp}^{opp} \to (\text{Set}), S \mapsto \{(X, \rho) \mid X/S + \rho : X \times F S \to X \times S \overline{S}\}/\cong.$$ 

Two pairs $(X, \rho), (X', \rho')$ are isomorphic if there exists an isomorphism $\gamma : X \to X'$ such that $\gamma \circ \rho = \rho'$.

Due to the Lemma of Drinfeld 4.17, objects $(X, \rho)$ as in Definition 5.5 have no automorphisms.

**Theorem 5.6** (Rapoport-Zink). The functor $N_X$ is representable by a formal scheme, locally formally of finite type over $\text{Spf} W(F)$.

**Definition 5.7.** Let $\rho : X \to Y$ be a quasi-isogeny of $p$-divisible groups over a connected scheme $S$. Then locally on $S$, there exists $N \in \mathbb{N}$ such that $p^N \rho$ is an isogeny. We define the height of $\rho$ by the relation

$$|\ker(p^N \rho)| = p^{ht(X)+ht(\rho)}.$$ 

The height is locally constant.

**Remark 5.8.** i) $N_X = \bigsqcup_{i \in \mathbb{Z}} N_{X,i}$ where $N_{X,i}$ is the locus where $\rho$ has height $i$.

ii) The group of quasi-isogenies $g : X \to X$ acts on $N$ via $(X, \rho) \mapsto (X, \rho \circ g^{-1})$.

iii) $N_X$ only depends on $X$ up to isogeny.

**Remark 5.9** (Important Remark). Let $(X, \rho) \in N(S)$ and let $s$ be a quasi-endomorphism of $X$. Then $p^{-1} \circ s \circ p$ is a quasi-endomorphism of $X \times S \overline{S}$. By Drinfeld rigidity 4.17, we can view it as quasi-endomorphism of $X$ itself.

By the unramifiedness of $\text{Hom} 4.16$, there exists a closed formal subscheme $Z(s) \subset N_X$ such that $(X, \rho) \in N_{X,i}$ if and only if $ps\rho^{-1}$ is an endomorphism of $X$. 

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5.4 The Drinfeld moduli problem

A good reference is [11].

We let $B_p := B \otimes \mathbb{Q}_p$ with maximal order $\mathcal{O}_{B_p} := \mathcal{O}_B \otimes \mathbb{Z}_p$. Our chosen base point $(A_0, \iota_0)$ yields a $p$-divisible group $X := A_0(p)$ together with an action $\iota_X : \mathcal{O}_{B_p} \to \text{End}(X)$. The group $X$ has height 4 and satisfies the special condition, i.e.

$$\forall b \in \mathcal{O}_{B_p} \quad \text{charpol}(\iota(b) \mid \text{Lie}(X))(T) = \text{charrd}(b)(T).$$

**Remark 5.10.** The pair $(X, \iota_X)/\mathbb{F}$ is unique up to $\mathcal{O}_{B_p}$-linear isogeny.

**Definition 5.11.** We define the Drinfeld moduli problem to be the following functor on $\text{Nilp}$:

$$\mathcal{N} : S \mapsto \{(X, \iota, \rho) \mid \ldots \}/\cong,$$

where $X/S$ is a $p$-divisible group of height 4, $\iota : \mathcal{O}_{B_p} \to \text{End}(X)$ an action of $\mathcal{O}_{B_p}$ such that the special condition is satisfied and $\rho$ a quasi-isogeny

$$\rho : X \times_S S \to X \times_S S.$$

**Lemma 5.12.** The functor $\mathcal{N}$ is representable by a formal scheme, locally formally of finite type over $\text{Spf} W$.

**Proof.** By Rapoport-Zink, the space of quasi-isogenies $\mathcal{N}_X$ is representable. The locus where the action of $\mathcal{O}_{B_p}$ lifts is a closed subfunctor and the special condition is also closed.

Let $B^{(p)}/\mathbb{Q}$ be the quaternion algebra with the same invariants as $B$ except at the places $p$ and $\infty$. In other words, $[B^{(p)}] + [B] = [D^{(p)}]$ in the Brauer group of $\mathbb{Q}$. Let $I := B^{(p)}/\mathbb{Q}$ be the algebraic group of units of $B^{(p)}$.

Then $I(\mathbb{Q}_p)$ be the group of automorphisms of $(X, \iota_X)$, i.e. the group of $\mathcal{O}_{B_p}$-linear quasi-isogenies $X \to X$. There is an isomorphism $I(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$. As in Remark 5.8, $I(\mathbb{Q}_p)$ operates on $\mathcal{N}$ by composition in the framing

$$g \cdot (X, \rho) := (X, \rho \circ g^{-1}).$$

There is a decomposition $\mathcal{N} = \bigcup_i \mathcal{N}_i$, where $\mathcal{N}_i$ is characterized by the fact that $\text{ht}(\rho|_{\mathcal{N}_i}) = 2i$. Note that $\text{ht}(\rho \circ g^{-1}) = \text{ht}(\rho) - 2e_p(\det g)$. So $I(\mathbb{Q}_p)$ acts transitively on the $\mathcal{N}_i$.

**Remark 5.13.** The space $\mathcal{N}$ plays the same role in the $p$-adic uniformization as the factor $GL_{2d}(\mathbb{Q}_p)/GL_{2d}(\mathbb{Z}_p)$.

5.5 The Drinfeld half-plane

The structure of $\mathcal{N}_0$ was determined by Drinfeld. It is isomorphic to the Drinfeld half-plane.

Let $T = T(PGL_2(\mathbb{Q}_p))$ be the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$. Its vertices are the homothety classes of lattices in $\mathbb{Q}_p^2$. Two vertices $[\Lambda_0]$ and $[\Lambda_1]$ are joined by an edge if there exist representatives $A_0$ and $A_1$ such that $A_0 \subset A_1$ of index 1.
For $[\Lambda] \in \mathcal{T}$ a vertex, we define
\[ \hat{\Omega}_{[\Lambda]} := (\mathbb{P}(\Lambda))^\wedge \setminus \mathbb{P}(\Lambda)(\mathbb{F}_p). \]
Here $\wedge$ denotes the $p$-adic completion of a scheme. The choice of a basis $\Lambda = (e_0, e_1)$ defines an isomorphism
\[ \hat{\Omega}_{[\Lambda]} \cong \text{Spf} \mathbb{Z}_p[T, (T^p - T)^{-1}]^\wedge. \]

For $\Delta = [\Lambda_0, \Lambda_1] \subset \mathcal{T}$ an edge, we define a chart $\hat{\Omega}_\Delta$. Let us do this in coordinates. Apply the elementary divisor theorem to get representatives and bases $\Lambda_0 = (e_0, e_1), \Lambda_1 = (pe_0, e_1)$. Then define
\[ \hat{\Omega}_\Delta = \text{Spf} \mathbb{Z}_p[T_0, T_1, (1 - T_0^{p-1}), (1 - T_1^{p-1})^{-1}]^\wedge / T_0T_1 - p. \]
This chart canonically contains $\hat{\Omega}_{\Lambda_0}$ and $\hat{\Omega}_{\Lambda_1}$.

**Definition 5.14.** The formal Drinfeld half-plane $\hat{\Omega}$ is the formal scheme which is defined by gluing \{ $\hat{\Omega}_\Delta \mid \Delta \subset \mathcal{T}$ simplex \} such that
\[ \hat{\Omega}_\Delta \cap \hat{\Omega}_{\Delta'} = \begin{cases} \hat{\Omega}_{\Delta \cap \Delta'} & \text{if } \Delta \cap \Delta' \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases} \]
Note that $\hat{\Omega}_\Delta \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a configuration of $\mathbb{P}^1$'s, intersecting transversally with dual graph equal to $\mathcal{T}$. Also note that $\hat{\Omega}$ is regular.

Note that $\text{PGL}_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ via $g[\Lambda] = [g\Lambda]$. And $g$ defines an isomorphism
\[ \mathbb{P}(\Lambda) \xrightarrow{\cong} \mathbb{P}(g\Lambda). \]
This defines an action of $\text{PGL}_2(\mathbb{Q}_p)$ on $\hat{\Omega}$. This action is very explicit! The stabilizer of an irreducible component $\mathbb{P}(\Lambda)$ equals $\text{PGL}(\Lambda)$. Fix a basis $\Lambda = (e_0, e_1)$ and let
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}(\Lambda). \]
The $g$ acts as Möbius transformation
\[ g : t = e_0/e_1 \mapsto \frac{at + b}{ct + d}. \]
Let $\hat{\Omega}^* := \mathbb{Z} \times \hat{\Omega}$ and endow it with an action of $\text{GL}_2(\mathbb{Q}_p)$ by the rule $g : [i, \mathbb{P}(\Lambda)] \mapsto [i - v_p(\det g), \mathbb{P}(g\Lambda)]$.

**Theorem 5.15.** Fix an isomorphism $I(\mathbb{Q}_p) \cong \text{GL}_2(\mathbb{Q})$. Then there is a $G$-equivariant isomorphism
\[ \mathcal{N} \cong \hat{\Omega}^*_W := \hat{\Omega}^* \times_{\mathbb{Z}_p} \text{Spf} W. \]

**Corollary 5.16.** $\mathcal{N}$ is regular of dimension 2 with semi-stable reduction over $\text{Spf} W$.

Here semi-stable is meant in the following sense: The special fiber $\mathcal{N} \otimes_W \mathbb{F}$ is a configuration of $\mathbb{P}^1$'s which intersect in ordinary double points and each $\mathbb{P}^1$ intersects at least three other $\mathbb{P}^1$'s.
5.6 \( p \)-adic Uniformization of \( \mathcal{M} \)

We have seen that \((A_0, \iota_0)\) is isogeneous to a product of supersingular elliptic curves. Thus \(\text{End}^0(A_0) \cong M_2(D(p))\) and by the centralizer theorem, \(\text{End}_{\mathbb{O}_p}(A_0) \cong B(p)\).

Recall that we defined \(I = B(p)_{\times}\) as algebraic group over \(\mathbb{Q}\). Then \(I(\mathbb{Q})\) is the group of \(\mathbb{O}_B\)-linear quasi-isogenies of \(A_0\). Furthermore, \(I(\mathfrak{a}^p) = \text{Aut}_B(\mathbb{W}(A_0))\) and we let \(I(\mathfrak{a}^p) \supset \mathcal{K} := \text{Stab}(\mathbb{T}(A_0))\). This is a maximal compact subgroup. If wanted, we could also identify \(B(p) \otimes \mathfrak{a}^p \cong B \otimes \mathfrak{a}^p\) such that \(\mathcal{K} = (\mathbb{O}_B \otimes \mathbb{Z}^p)_{\times}\).

Now consider the space \(\hat{\Omega}_W \times I(\mathfrak{a}^p)\). To any \(S\)-valued point \([\tau, \iota, \rho, g]\) we associate an abelian surface with \(\mathbb{O}_B\)-action over \(S\) as follows. First consider \(A_0 \times_S \hat{S}\). The datum of \(\rho\) and \(g\) define an isogeny \(\alpha : A_0 \times \hat{S} \rightarrow \hat{A}\) as in (5.1). Then \(b \mapsto \alpha(b)\alpha^{-1} \in \text{End}^0(\hat{A})\) defines an action of \(\mathbb{O}_B\) on \(\hat{A}\). (I.e. these are not only quasi-endomorphisms.) To check this, we have to check that for all \(b \in \mathbb{O}_B\), \(\alpha(b)\alpha^{-1}\) is an endomorphism of \(\hat{A}(\mathbb{Q})\) and preserves the Tate module \(\mathbb{T}(\hat{A})\). But these conditions follow from the \(\mathbb{O}_B\)-linearity of \(\rho\) and \(g\).

Now \(X\) is a deformation of \(X \times_S \hat{S}\) to which the action of \(\mathbb{O}_B\) lifts. By the theorem of Serre, there is a corresponding deformation \(A\) of \(\hat{A}\) to which the action of \(\mathbb{O}_B\) lifts. Thus we defined a transformation of functors

\[
\hat{\Omega}_W \times I(\mathfrak{a}^p) / \mathcal{K} \rightarrow \mathcal{M} |_{\text{Nilp}_W}
\]

\([\tau, \iota, \rho, g] \mapsto [\alpha : A_0 \times \hat{S} \rightarrow \hat{A}, \iota] \mapsto (A, \iota).
\]

Note that the restriction \(\mathcal{M} |_{\text{Nilp}_W}\) is nothing but the \(p\)-adic completion \(\mathcal{M}_p = \lim \mathcal{M} \otimes W/p^n\). There is an obvious diagonal action of \(I(\mathbb{Q})\) on the left hand side of (5.2) and the morphism factors over the quotient. So we almost proved the following theorem.

**Theorem 5.17 (Cherednik-Drinfeld).** There is an isomorphism of formal stacks

\[
[I(\mathbb{Q}) \setminus \hat{\Omega}_W] \times I(\mathfrak{a}^p)/\mathcal{K} \cong \mathcal{M}_p.
\]

**Corollary 18.** The Shimura curve \(\mathcal{M}\) is a regular surface. The geometric fibres of \(\mathcal{M} \rightarrow \mathbb{Z}\) are semi-stable and connected.

Note that the irreducible components of \(\hat{\Omega}_W\) are in bijection with \(GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p) = I(\mathbb{Q}_p)/\mathcal{K}_p\).

**Corollary 5.19.** There is a bijection

\[
I(\mathbb{Q}) \setminus I(\mathfrak{a}^p)/\mathcal{K}_p \cong \{\text{irred comp of } \mathcal{M}_p\}.
\]

This set can be identified with the class group of \(B(p)\).

6 Structure of Special Cycles at \(p \mid D(B)\)

Still, the reference is [11].

As in the complex situation, we also uniformize the \((p\text{-adic completions}) \mathbb{Z}(t)_p\).
6.1 Special Cycles on $\hat{\Omega}^*$

Recall that $X = A_0(p)$ with its action by $O_{B_p} = O_B \otimes Z_p$. Let $V(X, \iota_X) := \{ j \in \text{End}_0^0(X) \mid \text{tr}(j) = 0 \}$ denote the special endomorphisms of $X$. Again we have the $\mathbb{Q}_p$-valued quadratic form $Q(j) = -j^2$. Recall that $\text{End}_0^0(X, \iota_X) \cong M_2(\mathbb{Q}_p)$, so $V(X, \iota_X) \cong \mathbb{M}_2(\mathbb{Q}_p)$. There is an inclusion $V(A_0, \iota_0) \subset V(X, \iota_X)$.

**Definition 6.1.** For $0 \neq j \in \text{End}_0^0(X, \iota_X)$, we define $Z(j) \hookrightarrow \mathcal{N}$ to be the locus on which $\rho j \rho^{-1}$ is an isogeny. If $j$ is a special endomorphism, then we call this a special cycle.

**Remark 6.2.**

i) Obviously, the cycle $Z(j)$ only depends on the $\mathbb{Z}_p$-algebra generated by $j$ in $\text{End}(X)$.

ii) The action of an element $g \in I(\mathbb{Q}_p)$ on $\mathcal{N}$ defines an isomorphism $g : Z(j) \longrightarrow Z(gjg^{-1})$.

For $g \in I(\mathbb{Q}_p)$, we denote by $\mathcal{N}^g$ the fixed point scheme of $g$. It is empty, if $\det(g) \notin \mathbb{Z}_p^\times$.

**Theorem 6.3.** Let $j \in V(X)$ with $0 \neq Q(j) = \varepsilon p^\alpha$ where $\varepsilon \in \mathbb{Z}_p^\times$. Then

$$Z(j) \cong \begin{cases} \emptyset & \text{if } \alpha < 0 \\ \mathcal{N}^j & \text{if } \alpha = 0 \\ \mathcal{N}^{1+j} & \text{if } \alpha > 0. \end{cases}$$

**Proof.** If $\alpha < 0$, then $j$ has negative height. Thus it cannot be an endomorphism of any $X$.

If $\alpha = 0$ and if $\rho j \rho^{-1}$ is an endomorphism of $X$, then it is even an automorphism of $X$. This is the case if and only if it defines an isomorphism $(X, \iota, \rho j) \cong (X, \iota, \rho)$ which is equivalent to $X \in \mathcal{N}^j$.

If $\alpha > 0$, then we use the equality $Z(j) = Z(1 + j)$. Now the previous argument did not need $\text{tr}(j) = 0$ and so also holds for $1 + j$.

6.2 Structure of $Z(j)$

Note the picture on page 9 in [11].

Now fix $j$ with $Q(j) \neq 0$ and $\alpha \geq 0$. Let $\mathcal{N}^{\text{ord}} = \hat{\Omega}_{W}^{\text{ord}}$ be the ordinary locus, i.e. the complement of the intersection points of the $\mathbb{P}^1$’s. Let $Z(j)^{\text{ord}} = Z(j) \cap \mathcal{N}^{\text{ord}}$.

In general, $Z(j)$ is purely 1-dimensional. But it is not a divisor (i.e. locally generated by one equation). It can have embedded components at the intersection points of the $\mathbb{P}^1$’s. But $Z(j)^{\text{ord}}$ is a divisor on $\hat{\Omega}_{W}^{\text{ord}}$. It can have up to two horizontal components. Computing everything is really immediate.

**Example 6.4.** Let $\Lambda = (e_0, e_1)$ be a lattice with basis, let

$$j = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \text{End}(\Lambda)$$
and define \( n \) by \( p^n = (a, b, c) \), assume \( \alpha > 0 \). Then

\[
\Omega_{\alpha}^{ord} \cap Z(j) = V(T - \frac{(1 + a)T + b}{eT + (1 - a)}) = V(p^n(c'T^2 - 2a'T + b')).
\]

So we see that \( \Omega_{\alpha}^{ord} \) occurs with multiplicity \( n \) in \( Z(j)^{ord} \). If there are horizontal components, these are determined by the second factor of the equation.

Performing a more careful analysis yields the next proposition. Let us write

\[
Z(j)^{ord, vert} = \sum_{\Lambda} m_{\Lambda}(j)\Omega_{\alpha}^{ord}.
\]

Let \( T(j) \) denote the fixed point set of \( j \) acting on \( T \).

**Proposition 6.5.** The \( m_{\Lambda}(j) \) can be determined on the Bruhat-Tits tree \( T \) as follows. Write \( Q(j) = \varepsilon p^n \) and define \( j = j/p^{\lfloor \alpha/2 \rfloor} \). If \( \alpha \) is even, then \( \varepsilon_p(Q(j)) = 0 \) and

\[
T(\tilde{j}) = \begin{cases} 
\text{vertex} \\
\text{apartment}
\end{cases}
\]

If \( \alpha \) is odd, then \( \varepsilon_p(Q(j)) = 1 \) and \( T(\tilde{j}) \) is an edge. Then \( T(j) \) is a ball of radius \( \lfloor \alpha/2 \rfloor \) around \( T(\tilde{j}) \). If \( \Lambda \in T(j) \) has distance \( i \) of \( T(\tilde{j}) \), then \( m_{\Lambda} = \lfloor \alpha/2 \rfloor - i \).

**Example 6.6.** If \( j = (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}) \), then it fixes the chain of (homothety classes) of lattices

\[
\ldots \subset (p^2 e_1, e_2) \subset (p e_1, e_2) \subset (e_1, e_2) \subset (p^{-1} e_1, e_2) \subset \ldots.
\]

Then \( Z(j) \) is the sum of the irreducible components corresponding to these lattices, each with multiplicity 1. There are no vertical components.

There is a picture illustrating the three cases in [11, p. 9].

6.3 \( p \)-adic uniformization of special cycles

Let me first describe \( Z(t)(\mathbb{F}) \) in terms of \( \Omega_{t}^{\ast}(\mathbb{F}) \times I(A_f^p)/K^p \).

Let \((A, \iota) \in M(\mathbb{F}) \) and choose an isogeny \( f : (A_0, \iota_0) \rightarrow (A, \iota) \). Then \( f^{-1}V(A, \iota)_{\mathbb{Q}}f = V(A_0, \iota_0)_{\mathbb{Q}} \). The group \( I(A_0) \) acts by conjugation (i.e. pull-back) on \( V(A_0, \iota_0)_{\mathbb{Q}} \):

\[
s^{*}x = sx s^{-1}.
\]

So set-theoretically,

\[
Z(t)(\mathbb{F})/\cong = \bigcup_{x \in I(A_0) \backslash V(A_0, \iota_0)_{\mathbb{Q}} | Q(x) = t} \{ A | \exists \text{ quasi-isogeny } f : A_0 \rightarrow A \text{ s.th. } fx f^{-1} \in \text{End}(A) \}/\cong.
\]

Now the condition \( fx f^{-1} \in \text{End}(A) \) can be checked on Tate modules. Let \( f : A_0 \rightarrow A \) correspond to \((\rho : \mathbb{X} \rightarrow X, gT^p(A) \subset \mathbb{P}^p(A)) \). Then \( fx f^{-1} \in \text{End}(A) \) if and only if \( \rho \in Z(x) \) and \( xgT^p(A_0) \subset gT^p(A_0) \) which is equivalent to \( g^{-1}xgT^p(A_0) \subset T^p(A_0) \).

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These ideas work in families and yield the $p$-adic uniformization of $Z(t)$. Let us define

$$C(t) := \prod_{x \in V(A_{0, t_0}) \cap Q(x) = t} Z(x) \times \{gK^p \mid g^{-1} x g \in K^p\} \twoheadrightarrow N \times I(h_f^p)/K^p.$$ 

The group $I(Q)$ acts diagonally on this space. Namely $s \in I(Q)$ defines an isomorphism

$$s : Z(x) \times \{gK^p \mid g^{-1} x g \in K^p\} \twoheadrightarrow Z(sxs^{-1}) \times \{gK^p \mid g^{-1} x s^{-1} g \in K^p\}$$

$$(p, gK^p) \mapsto (\rho^{-1}, sgK^p).$$

**Proposition 6.7.** There is an isomorphism of formal stacks

$$[I(Q) \setminus C(t)] \cong Z(t)_p$$

which is compatible with the uniformization of $M_p$.

7 Outlook: Modularity of the Vertical component

The reference is [14][Chap. 4.3].

We now sketch the proof of the modularity of the vertical component of $\hat{\phi}$. Let $Y \subset \mathcal{M}$ be an irreducible component in a fiber of bad reduction. For more details, we refer to [14, 4.3].

**Proposition 7.1.** The series $\langle \hat{\phi}, (Y, 0) \rangle = \langle \hat{Z}(0, v), (Y, 0) \rangle + \log(p), \sum_{t > 0} (Z(t), Y) q^t \in \mathbb{C}[[q]]$ is the $q$-expansion of a holomorphic modular form of weight $3/2$.

**Proof.** We will identify $\sum_{t \geq 0} (\hat{Z}(t, v), (Y, 0))$ with a definite theta series of weight $3/2$.

Let $V' := V(A_{0, t_0}) \otimes \mathbb{Q}$ with lattice $L := V(A_{0, t_0})$. Then $V'$ is a 3-dimensional quadratic subspace of $(B^p, N \mathfrak{m})$, so it is definite.

Let $[\hat{Y}, g] \in \hat{\Omega}_{Y'}^* \times I(A_f^p)/K^p$ be an irreducible component mapping to $Y \otimes \mathbb{F}$. Then

$$(Y, Z(t))_{\mathcal{M}} = (Y \otimes \mathbb{F}, Z(t)_p)_{\mathcal{M}_p} = ([\hat{Y}, g], C(t))_{\hat{\Omega}_{Y'}^* \times I(A_f^p)/K^p}$$

In the last expression, only those components of $C(t)$ play a role, which lie over $g \in I(A_f^p)/K^p$. These correspond to $x \in V'$ with $Q(x) = t$ and $g^{-1} x g \in L \otimes \mathbb{Z} \hat{\mathbb{Z}}$. Define the characteristic function $\varphi^p := 1_{g(L \otimes \mathbb{Z} \hat{\mathbb{Z}})} \in C_c^\infty(V'(\mathbb{Q}_p))$.

**Lemma 7.2** (see [14, Proposition 4.3.2]). There exists a function $\mu_{\varphi} \in C_c^\infty(V'(\mathbb{Q}_p))$ such that for all $x \in V'(\mathbb{Q}_p)$,

$$\mu_{\varphi}(x) = ([\hat{Y}, Z(x)]_{\hat{\Omega}_{Y'}^*}).$$

Performing an extra analysis of $\langle \hat{Z}(0, v), (Y, 0) \rangle$ and $\mu(0)$ yields that

$$\sum_{t \geq 0} (\hat{Z}(t, v), (Y, 0)) = \sum_{x \in V'(\mathbb{Q})} (\varphi^p \otimes \mu_{\varphi})(x) q^{Q(x)}.$$ 

The result follows. 

\[\square\]
References


