# The Geometry of Shimura Curves and special cycles

Andreas Mihatsch*

April 5, 2015

These are notes for the Alpbach pre-workshop in Zürich in April 2015. They contain some more details than the talks themselves.

## 1 Introduction

The reference for the introduction is [14][Chap. 1].

Let $B/\mathbb{Q}$ be an indefinite quaternion algebra, i.e. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$. Let

$$D := D(B) := \prod_{p \text{ s.th. } B\otimes\mathbb{Q}_p \text{Div.alg}} p$$

denote the discriminant of $B$. We fix a maximal order $\mathcal{O}_B$ of $B$. It is unique up to conjugacy.

**Definition 1.1.** We define the *Shimura curve* $\mathcal{M} \longrightarrow \operatorname{Spec} \mathbb{Z}$ associated to $B$ to be the following stack:

$$\mathcal{M}(S) = \{(A, \iota) \mid A \text{ abelian surface}/S + \iota : \mathcal{O}_B \longrightarrow \operatorname{End}(A) + \text{special condition}\}.$$

Special condition:

$$\operatorname{charpol}(\iota(b) \mid \operatorname{Lie} A)(T) = \operatorname{charred}(b)(T) \in \mathcal{O}_S[T] \quad \forall b \in \mathcal{O}_B.$$

The polynomial on the right is the reduced characteristic polynomial of $x$. If you are not familiar with the definition of a stack, just think of $\mathcal{M}(S)$ as the isomorphism classes of pairs $(A, \iota)$ together with their automorphisms.

**Remark 1.2.** i) We take it for granted that $\mathcal{M}$ is a Deligne-Mumford stack, of finite type over $\mathbb{Z}$.
ii) If $B = M_2(\mathbb{Q})$, then we can assume $\mathcal{O}_B = M_2(\mathbb{Z})$. Then $E \mapsto E \times E$ and $(A, \iota) \mapsto \ker\left(\left(\ {}_1\ \right)\right)$ induce an equivalence

$$Ell(S) := \{\text{Elliptic Curves}/S\} \cong \mathcal{M}(S).$$

So $\mathcal{M}$ is the usual modular curve.
iii) The special condition is closed and automatic in characteristic 0. It ensures flatness of $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$.

We will prove later in the course:

*mihatsch@math.uni-bonn.de

**Proposition 1.3.** *The DM-stack $\mathcal{M}$ is regular of dimension 2.*
*The morphism $\mathcal{M} \longrightarrow \operatorname{Spec}\mathbb{Z}$ is flat of relative dimension 1, smooth over $\operatorname{Spec}\mathbb{Z}[D(B)^{-1}]$ and has semi-stable reduction at $p \mid D(B)$. Its fibers are geometrically connected.*
*If $B$ is a division algebra, then $\mathcal{M}$ is proper over $\operatorname{Spec}\mathbb{Z}$.*

The stack $\mathcal{M}$ is a regular surface, so it has a good intersection theory. Also, $\mathcal{M}$ is canonically endowed with a whole family of cycles. Let us describe them in the language of Shimura varieties.

## 1.1 CM cycles on $\mathcal{M}$

Let $C := \operatorname{End}_B(B)$ and $G := C^\times$ as algebraic group over $\mathbb{Q}$. Note that $C = B^{op} \cong B$, but it is better to separate these algebras. Let $\mathcal{O}_C = \operatorname{End}_{\mathcal{O}_B}(\mathcal{O}_B)$ which is a maximal order in $C$. It yields $(\mathcal{O}_C \otimes \hat{\mathbb{Z}})^\times =: K \subset G(\mathbb{A}_f)$ and there is an isomorphism

$$\mathcal{M}_\mathbb{C} \cong [G(\mathbb{Q})\backslash\mathfrak{h}^\pm \times G(\mathbb{A}_f)/K].$$

Here $G(\mathbb{Q})$ acts on $\mathfrak{h}^\pm := \mathbb{C} \setminus \mathbb{R}$ via the choice of an isomorphism $G(\mathbb{R}) \cong GL_2(\mathbb{R})$.

Let $E \subset \mathbb{C}$ be an imaginary quadratic field with embedding $\rho : E \hookrightarrow C$. Let $T := E^\times$ be the corresponding torus with embedding $\rho : T \longrightarrow G$. This defines a divisor (Heegner divisor)

$$[T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\rho^{-1}(K)] \to \mathcal{M}_\mathbb{C}$$

on the Shimura curve.

## 1.2 Special cycles

When trying to extend these cycles to $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$, it is better to use a moduli theoretic approach. The *special cycles* are weighted combinations of integral models of these CM cycles.

**Definition 1.4.** Let $(A, \iota) \in \mathcal{M}(S)$. The space of *special endomorphisms* of $(A, \iota)$ is the $\mathbb{Z}$-module

$$V(A, \iota) = \{x \in \operatorname{End}_S(A, \iota) \mid tr(x) = 0\}.$$

It is a quadratic space with form

$$-x^2 = Q(x) \cdot \operatorname{id}_A.$$

It follows from the classification of $\operatorname{End}(A, \iota) \otimes \mathbb{Q}$ that this form is positive definite.

**Definition 1.5.** Let $t > 0$ be an integer. Define the *special cycle* $\mathcal{Z}(t) \longrightarrow \mathcal{M}$ as the moduli stack of $(A, \iota, x)$ with $Q(x) = t$.

The forgetful morphism $pr : \mathcal{Z}(t) \longrightarrow \mathcal{M}$ is unramified and finite but it is not a closed immersion. The image of $pr$ are the $(A, \iota)$ which admit complex multiplication by $\mathbb{Z}[\sqrt{-t}]$. The complex fiber $\mathcal{Z}(t)_\mathbb{C}$ is a linear combination of CM cycles as above.

## 1.3 Relation to modular forms

We assume now that $B$ is division such that $\mathcal{M}$ is proper. The generic fiber $\mathcal{Z}(t)_{\mathbb{C}}$ is reduced and 0-dimensional, so it is a finite union of points, each with a finite group of automorphisms. Its degree is defined as

$$\deg \mathcal{Z}(t)_{\mathbb{C}} = \sum_{(A,\iota,x)\in\mathcal{Z}(t)(\mathbb{C})} \frac{1}{|\mathrm{Aut}(A,\iota,x)|}.$$

The index set of the special cycles is the same as that of elliptic modular forms. We assemble them into a generating series

$$\phi_{\deg}(\tau) := -\mathrm{vol}(\mathcal{M}(\mathbb{C})) + \sum_{t>0} \deg(\mathcal{Z}(t)_{\mathbb{C}})q^t \ \in \mathbb{C}[[q]].$$

The motivation for the constant term is that purely formally, $\mathcal{Z}(0) = \mathcal{M}$. Also, this volume is known,

$$\mathrm{vol}(\mathcal{M}(\mathbb{C})) = \frac{1}{12} \prod_{p|D(B)} (p-1).$$

**Proposition 1.6.** *The series $\phi_{\deg}$ is the $q$-expansion of a holomorphic modular form of weight $3/2$ and level $\Gamma_0(4D(B))$.*

The proof of this proposition is not via general arguments. Instead one explicitly computes all degrees $\deg \mathcal{Z}(t)$ and compares them to the Fourier coefficients of a known Eisenstein series! In this course, I will only explain the computation of the degrees.

## 1.4 Arithmetic special cycles

The reference for Chow groups and intersection theory is [14][Chap. 2].

Note that $\mathcal{M} \longrightarrow \mathrm{Spec}\,\mathbb{Z}$ is flat and proper, but $\mathrm{Spec}\,\mathbb{Z}$ is not "compact" itself. We use Arakelov theory to define arithmetic cycles $\hat{\mathcal{Z}}(t,v)$.

**Definition 1.7.** i) Let $Z^1_{\mathbb{R}}(\mathcal{M})$ be the group of divisors on $\mathcal{M}$ with coefficients in $\mathbb{R}$. In other words, $Z^1_{\mathbb{R}}(\mathcal{M})$ is the $\mathbb{R}$-vector space generated by the irreducible closed substacks $\mathcal{Z} \hookrightarrow \mathcal{M}$.
ii) Let $\mathcal{Z} \in Z^1_{\mathbb{R}}(\mathcal{M})$ and denote by $\delta_{\mathcal{Z}} : C^\infty(\mathcal{M}(\mathbb{C})) \longrightarrow \mathbb{C}$ its $\delta$-distribution. A *Green function* for $\mathcal{Z}$ is a smooth (real-valued) function $\Xi(v)$ on $\mathcal{M}(\mathbb{C})$ with logarithmic growth along $\mathcal{Z}_{\mathbb{C}}$ such that the *Green equation* holds:

$$dd^c\Xi(v) + \delta_{\mathcal{Z}} = [\omega]$$

for some smooth $(1,1)$-form $\omega$. We let $\hat{Z}^1_{\mathbb{R}}(\mathcal{M})$ denote the $\mathbb{R}$-vector space of *Arakelov divisors*, i.e. of pairs $(\mathcal{Z}, \Xi(v))$, where $\Xi(v)$ is a Green function for $\mathcal{Z}$.
iii) To a meromorphic function $f \in \mathbb{Q}(\mathcal{M})^\times$, we associate the *principal Arakelov divisor*

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f_{\mathbb{C}}|^2).$$

iv) The first arithmetic Chow group (with real coefficients) $\widehat{CH}^1(\mathcal{M})$ is the quotient of $\hat{Z}^1_{\mathbb{R}}$ by the $\mathbb{R}$-vector space generated by the principal Arakelov divisors.

**Remark 1.8.** Let $\mathcal{L}$ be a line bundle on $\mathcal{M}$ together with a smooth metric $|\cdot|$ on the complex fiber $\mathcal{L}_{\mathbb{C}}$. Let $s \in \mathbb{Q}(\mathcal{L})^{\times}$ be a meromorphic section. Then $(\mathrm{div}(s), -\log|s_{\mathbb{C}}|^2)$ is an Arakelov divisor and every Arakelov divisor with *integral* coefficients arises in this way.

In his Annals paper, Kudla defines certain functions $\Xi(t,v)$ for $t \in \mathbb{Z} \setminus \{0\}$, which are Green functions for $\mathcal{Z}(t)$ if $t > 0$ and smooth if $t < 0$. For the definition, see [14, Chap. 3.5]. This defines classes in $\widehat{CH}^1(\mathcal{M})$,

$$\hat{\mathcal{Z}}(t,v) := \begin{cases} (\mathcal{Z}(t), \Xi(t,v)) & \text{if } t > 0 \\ (0, \Xi(t,v)) & \text{if } t < 0 \\ \text{see KRY (4.2.4)} & \text{if } t = 0. \end{cases}$$

We form the formal generating series

$$\widehat{\phi} := \sum_{t \in \mathbb{Z}} \widehat{\mathcal{Z}}(t,v) q^t \ \in \widehat{CH}^1(\mathcal{M})[[q, q^{-1}]].$$

Then the main result is the following theorem.

**Theorem 1.9.** *For $\tau = u + iv$, the series $\widehat{\phi}(\tau)$ is a (nonholomorphic) modular form of weight $3/2$ and level $\Gamma(4D(B))$ with values in $\widehat{CH}^1(\mathcal{M})$.*

This is Theorem A in [14]. See the introduction there for the precise meaning of this statement.

*About the proof:* The first Chow group $\widehat{CH}^1(\mathcal{M})$ is endowed with an intersection product (see [14][Chap. 2])

$$\langle \ , \ \rangle : \widehat{CH}^1(\mathcal{M}) \times \widehat{CH}^1(\mathcal{M}) \longrightarrow \mathbb{R}.$$

There is a decomposition, orthogonal for $\langle \ , \ \rangle$,

$$\widehat{CH}^1(\mathcal{M}) = \widetilde{\mathrm{MW}} \oplus (\mathbb{R}\hat{\omega} \oplus \mathrm{Vert}) \oplus C^{\infty}(\mathcal{M}(\mathbb{C}))_0.$$

Here, $C^{\infty}(\mathcal{M}(\mathbb{C}))_0$ are the functions orthogonal to the constant ones. And Vert is the subspace of $\widehat{CH}^1(\mathcal{M})$ generated by divisors $(Y, 0)$, where $Y$ is an irreducible component of some fiber of $\mathcal{M}/\mathbb{Z}$. The class $\hat{\omega}$ is the class of the metrized Hodge bundle. The Mordell-Weil space $\widetilde{\mathrm{MW}}$ is then the orthogonal complement of the other summands. Restriction to the generic fiber defines an isomorphism

$$\widetilde{\mathrm{MW}} \cong \mathrm{Jac}(\mathcal{M}_{\mathbb{Q}})(\mathbb{Q}) \otimes \mathbb{R}.$$

Let $\hat{\phi}^{\infty}$ be the components of $\hat{\phi}$ in $C^{\infty}(\mathcal{M}(\mathbb{C}))_0$ and let $\hat{\phi}^0$ be the component in the other summands. For the modularity of $\hat{\phi}^{\infty}$, see [14, 4.4].

To prove the modularity of $\hat{\phi}^0$, ones show that for every $\mathcal{Z} \in \widetilde{\mathrm{MW}} \oplus (\mathbb{R}\hat{\omega} \oplus \mathrm{Vert})$, the series $\langle \hat{\phi}^0, \mathcal{Z} \rangle$ is a modular form. This is done by "explicitly" computing its coefficients $\langle \hat{\mathcal{Z}}(t,v), \mathcal{Z} \rangle$. These coefficients can then be compared to the coefficients of "known" modular forms, usually Eisenstein series or Theta series. For details, see [14][Chap. 4]. In particular, there are no "abstract" arguments which yield the modularity of $\hat{\phi}$. □

**Remark 1.10.** The degree of the generic fiber

$$(\mathcal{Z}, \Xi) \mapsto \deg(\mathcal{Z}_{\mathbb{Q}})$$

defines a linear form on $\widehat{CH}^1(\mathcal{M})$. Thus the modularity of $\phi_{\deg}$ follows from the modularity of $\hat{\phi}$.

## 1.5 The vertical component

The reference is [14][Chap. 4.3].

In this course, we will focus on the series $\langle \hat{\phi}, \mathcal{Z} \rangle$, where $\mathcal{Z} \in \mathrm{Vert}$ is a vertical divisor.

Recall that $\mathcal{M} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is a proper flat relative curve with geometrically connected fibers. Let $\mathrm{Vert} \subset \widehat{CH}^1(\mathcal{M})$ be the subspace generated by the cycles of the form $(Y, 0)$, where $Y$ is a linear combination of irreducible components of fibers. The main result of this course is

**Proposition 1.11.** *Let $\mathcal{Z} \in \mathrm{Vert}$ a vertical Arakelov divisor. Then*

$$\langle \hat{\phi}, \mathcal{Z} \rangle$$

*is a (holomorphic) modular form of weight $3/2$.*

It suffices to prove this for irreducible divisors $\mathcal{Z} = (Y, 0)$ where $Y$ is a vertical irreducible component. Let $p$ be the prime below $Y$. I did not give the definition of $\langle\ ,\ \rangle$, it is enough to know for us that

$$\langle (\mathcal{Z}, \Xi), (Y, 0) \rangle = \log(p) \cdot (\mathcal{Z}, Y).$$

Here the intersection number $(\mathcal{Z}, Y)$ is defined as usual if $\mathcal{Z}$ and $Y$ have no common components. The product $(Y, Y)$ is defined as follows: Write $V(p) = aY + R \in Z^1_{\mathbb{R}}(\mathcal{M})$ and define $(Y, Y) := -\frac{1}{a}(R, \mathcal{Z})$.

Assume that $p \nmid D(B)$, i.e. assume that $\mathcal{M}$ is smooth over $p$. We will see:

**Proposition 1.12.** *The cycle $\mathcal{Z}(t)[D(B)^{-1}]$ is a divisor. It is flat over* $\operatorname{Spec} \mathbb{Z}[D(B)^{-1}]$.

It follows from the proposition that $\mathcal{Z}(t)$ and $Y$ have no common components. Thus,

$$\langle \hat{\mathcal{Z}}(t, v), (Y, 0) \rangle = \log(p) \cdot (\mathcal{Z}(t), Y) = \begin{cases} \log(p) \cdot \deg(\mathcal{Z}(t)_{\mathbb{C}}) & \text{if } t > 0 \\ -\log(p) \mathrm{Vol}(\mathcal{M}(\mathbb{C})) & \text{if } t = 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Thus $\langle \hat{\phi}, (Y, 0) \rangle = \log(p) \cdot \phi_{\deg}$. So modularity will follow from that of $\phi_{\deg}$.

If $Y$ is an irreducible component over $p \mid D(B)$, the computation of $\langle \mathcal{Z}(t), Y \rangle$ is more difficult and relies on the $p$-adic uniformization of the fiber at $p$. It is the principal aim of this course to explain the geometry of $\mathcal{M}$ at $p \mid D(B)$ and thereby prove the above proposition.

# Contents

### 1.6 Further remarks

i) By global class field theory

$$\{\text{Indefinite Quaternion Algebras}/\mathbb{Q}\} \cong \{\text{Even numbers of primes}\}$$
$$B \mapsto D(B)$$

The case $D(B) = 1$ corresponds to $B = M_2(\mathbb{Q})$. All maximal orders are conjugate.

ii) We refer to Deligne-Mumford [1] or Laumon-Moret Bailly for details concerning stacks.

iii) *Presentation of $\mathcal{M}$:*
Fix an integer $N$. A *level-$N$* structure on a pair $(A, \iota)$ is the choice of an $\mathcal{O}_B$-linear isomorphism

$$\eta : \mathcal{O}_B/N\mathcal{O}_B \cong A[N].$$

Define the stack $\mathcal{M}_N/\operatorname{Spec}\mathbb{Z}[N^{-1}]$ as

$$\mathcal{M}_N(S) = \{(A, \iota) \in \mathcal{M}(S) + \eta \text{ level-N structure for } (A, \iota)\}.$$

Then $\mathcal{M}_N \longrightarrow \mathcal{M}[N^{-1}]$ is a $(\mathcal{O}_B/N\mathcal{O}_B)^{\times}$-torsor. If $N \gg 0$, then $\mathcal{M}_N$ is a scheme. Thus $\mathcal{M}$ is locally (on $\operatorname{Spec}\mathbb{Z}$) the quotient of a scheme by a finite group.

## 2 The degree of special cycles

### 2.1 Uniformization of $\mathcal{M}_{\mathbb{C}}$

We now want to determine the geometric generic fiber $\mathcal{M}_{\mathbb{C}}$. Properties of $\mathcal{M}_{\mathbb{C}}$ will extend to almost all fiber of $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$. We start with some preparations.

*Preparations:* The algebra $B$ is endowed with the canonical *main involution* $b \mapsto b^{\tau}$. It is characterized by either of the following properties

- $b \cdot b^{\tau} = \operatorname{Nrd}(b)$.
- $b + b^{\tau} = \operatorname{trd}(b)$.
- $(T - b)(T - b^{\tau}) = \operatorname{chard}(b)(T)$.

- For all quadratic extensions $E/\mathbb{Q}$ and all embeddings $E \subset B$, $\tau|_E = \sigma_{E/\mathbb{Q}}$.

Thus $\tau \otimes \mathrm{id}_{\mathbb{R}}$ equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let $\delta \in \mathcal{O}_B$ be such that $\delta^\tau = -\delta$ and $\delta^2 < 0$ and define

$$b^* := \delta b^\tau \delta^{-1}.$$

Then $*$ is a positive involution, i.e. $\mathrm{tr}(xx^*) > 0$ for all $0 \neq x \in M_2(\mathbb{R})$.

On $B$ we define the $\mathbb{Q}$-valued non-degenerate alternating form

$$(x, y) := \mathrm{tr}(y^\tau \delta x).$$

Recall that $C = \mathrm{End}_B(B)$ and $G = C^\times$. Obviously $G = GSp_B(B, (\ ,\ ))$, i.e. all $B$-linear automorphisms of $B$ preserve the form $(\ ,\ )$ up to a constant.

**Lemma 2.1.** *Let $h : \mathbb{C} \longrightarrow C \otimes \mathbb{R}$ be an $\mathcal{O}_B$-linear complex structure on $B \otimes \mathbb{R}$. Then $B \otimes \mathbb{R}/\mathcal{O}_B$ is an abelian surface (i.e. projective as complex variety). Conversely, any $(A, \iota) \in \mathcal{M}(\mathbb{C})$ is of this form.*

Recall the following theorem about the projectivity of complex tori.

**Theorem 2.2** (Riemann)**.** *Let $T := \mathbb{C}^g/\Lambda$ be a complex torus. Then $T$ is projective if and only if there exists a non-degenerate, alternating pairing $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ with the following properties.*

- *It is compatible with the complex structure, i.e. $(i\cdot\ , i\cdot\ ) = (\ ,\ )$.*
- *The symmetric pairing $(\ , i\cdot\ )$ is positive definite.*

*Proof of the lemma.* We apply this to our situation. The identity $G = C^\times$ implies that $h(i) \in G(\mathbb{R})$ is compatible with $(\ ,\ )$, i.e. satisfies $(\ ,\ ) = (h(i)\ , h(i)\ )$. One can check that $(\ , h(i)\ )$ is definite so that there exists a polarization.

We now check that all points in $\mathcal{M}(\mathbb{C})$ are of this form. Let $\mathcal{O}_B \circlearrowright \mathbb{C}^2/\Lambda$ be a complex torus with action of $\mathcal{O}_B$. Then $\mathcal{O}_B$ acts on $\Lambda$.

**Lemma 2.3.** *The algebra $\mathcal{O}_B$ has class number one, i.e. every $\mathcal{O}_B$-module which is free of rank 4 as $\mathbb{Z}$-module is free of rank one over $\mathcal{O}_B$.*

Let us fix an identification $\Lambda \cong \mathcal{O}_B$. This identifies $\mathbb{C}^2 \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with $B \otimes_{\mathbb{Q}} \mathbb{R}$. The complex structure on $\mathbb{C}^2$ now arises from an $h : \mathbb{C} \longrightarrow C$ and thus defines an abelian variety. □

By Skolem-Noether, all homomorphisms $h : \mathbb{C} \longrightarrow C \otimes \mathbb{R}$ are conjugate. Let us fix an isomorphism $C \otimes \mathbb{R} \cong M_2(\mathbb{R})$. Then it is well-known that the conjugacy class $h$'s can be identified with $\mathfrak{h}^\pm = \mathbb{C} \setminus \mathbb{R}$. The action of $(C \otimes \mathbb{R})^\times \cong GL_2(\mathbb{R})$ on $\mathfrak{h}^\pm$ is then via Möbius transformations

**Proposition 2.4.** *Let $\Gamma := \mathcal{O}_C^\times \subset G(\mathbb{R})$. Then there is an isomorphism of orbifolds*

$$[\Gamma \backslash \mathfrak{h}^\pm] \longrightarrow \mathcal{M}_{\mathbb{C}}^{\mathrm{an}}.$$

*Sketch of proof.* Lemma 2.1 essentially yields a bijection $\Gamma\backslash\mathfrak{h}^{\pm} \cong \mathcal{M}(\mathbb{C})$. Now one can construct the abelian surface $((B \otimes_{\mathbb{Q}} \mathbb{R})/\mathcal{O}_B, h)$ in a family over $\mathfrak{h}^{\pm}$ which yields a holomorphic map

$$\mathfrak{h}^{\pm} \longrightarrow \mathcal{M}_{\mathbb{C}}^{an}.$$

One can check that this yields an isomorphism of orbifolds. □

**Corollary 2.5.** *The generic fiber of $\mathcal{M}$ is a smooth curve, geometrically connected. If $B$ is division, then this curve is projective.*

*Proof.* Connectedness follows since $\Gamma$ contains elements with negative discriminant which interchange the two half-planes of $\mathfrak{h}^{\pm}$. Projectivity since $\Gamma$ has no cusps if $B \ncong M_2(\mathbb{Q})$. □

**Remark 2.6.** The moduli problem with non-trivial level-structure (over $\mathbb{C}$) is not necessarily connected. It is Lemma 2.3 which fails when incorporating Level structure.

## 2.2 Uniformization of $\mathcal{Z}(t)_{\mathbb{C}}$

Let $(A, \iota) \in \mathcal{M}(\mathbb{C})$. The choice of an isomorphism $A = (B \otimes \mathbb{R})/\mathcal{O}_B$ yields a point $h \in \mathfrak{h}^{\pm}$ and identifies

$$\mathrm{End}(A, \iota) = \mathrm{Cent}_h(\mathcal{O}_C).$$

Define $L := \{x \in \mathcal{O}_C \mid \mathrm{tr}(x) = x^{\tau} + x = 0\}$ and $L(t) := \{x \in L \mid Q(x) = -x^2 = x^{\tau}x = t\}$. For $x \in L(t)$, let

$$D_x := \{h \mid xh = hx\}.$$

Then $|D_x| = 2$. Namely let $k := \mathbb{Q}(\sqrt{-t})$. Then $x$ defines an embedding

$$i_x : k \hookrightarrow C$$

such that $\mathbb{Z}[\sqrt{-t}] \subset i_x^{-1}(\mathcal{O}_C)$. But the centralizer of $k \subset C$ is $k$ itself. So $D_x = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, k \otimes \mathbb{R})$. We define

$$D_{\mathcal{Z}(t)} := \bigsqcup_{x \in L(t)} D_x \subset \mathfrak{h}^{\pm}.$$

The set $D_{\mathcal{Z}(t)}$ is stable under the action of $\Gamma$. More precisely, $\gamma \in \Gamma$ induces an isomorphism $\gamma : D_x \longrightarrow D_{\gamma x \gamma^{-1}}$. There is a $2:1$-surjection

$$D_{\mathcal{Z}(t)} \twoheadrightarrow L(t)$$

which is $\Gamma$-equivariant. Here $\Gamma$ acts by conjugation on $L(t)$.

**Proposition 2.7.** *There is an isomorphism of orbifolds*

$$[\Gamma\backslash D_{\mathcal{Z}(t)}] \xrightarrow{\cong} \mathcal{Z}(t)_{\mathbb{C}}.$$

Of course, this isomorphism is compatible with the uniformization with of $\mathcal{M}_{\mathbb{C}}$.

## 2.3 The degree of $\mathcal{Z}(t)_{\mathbb{C}}$

The reference is Part III of [13].

To state the result, we need some notation. Let $k = \mathbb{Q}(\sqrt{-t})$ have discriminant $d$. The order $\mathbb{Z}[\sqrt{-t}]$ has discriminant $4t = n^2 d$ where $n$ denotes its conductor. In general, we denot by $\mathcal{O}_{c^2d} \subset k$ the order of conductor $c$. We let $h(c^2d)$ be its class number and $w(c^2d) := |\mathcal{O}_{c^2d}^{\times}|$ be the number of roots of unity in $\mathcal{O}_{c^2d}$.

We define

$$H_0(t, D) := \sum_{c|n,(c,D)=1} \frac{h(c^2d)}{w(c^2d)}.$$

This is a variant of the Hurwitz class number, counting certain isomorphism classes of quadratic lattices, weighted with the number of their automorphisms.

We also define

$$\delta(d, D) := \prod_{l|D} (1 - \chi_d(l))$$

where $\chi_d$ denotes the quadratic character associated to $k$, i.e.

$$\chi_d(l) = \begin{cases} 1 & \text{if } l \text{ split in } k \\ -1 & \text{if } l \text{ inert in } k \\ 0 & \text{if } l \text{ ramified in } k. \end{cases}$$

Then $\delta(d, D) \neq 0$ if and only if there exists an embedding $k \hookrightarrow B$.

**Proposition 2.8.** *Let $k := \mathbb{Q}(\sqrt{-t})$.*
*There is an equality*

$$\deg \mathcal{Z}(t)_{\mathbb{C}} = 2\delta(d, D) H_0(t, D).$$

*In particular, $\mathcal{Z}(t)_{\mathbb{C}} = \emptyset$ if $k$ does not embed into $B$.*

*Proof.* It is clear that $k$ embeds into $B$ if and only if $\mathcal{Z}(t)_{\mathbb{C}} \neq 0$.

By the explicit description of $\mathcal{Z}(t)_{\mathbb{C}}$,

$$\deg \mathcal{Z}(t)_{\mathbb{C}} = 2 \sum_{x \in (L(t))/\Gamma} \frac{1}{|\Gamma_x|}.$$

Here the 2 is due to the fact that $D_{\mathcal{Z}(t)}$ is $2:1$ over $L(t)$. As mentioned above, each $x$ defines $i_x : k \longrightarrow C$ such that $\mathbb{Z}[\sqrt{-t}] \subset i_x^{-1}(\mathcal{O}_C)$. Let $c$ be the conductor of $i_x^{-1}(\mathcal{O}_C) = \mathcal{O}_{c^2d}$. Then $c \mid n$ and $(c, D) = 1$ since the order $i_x^{-1}(\mathcal{O}_C)$ is maximal at all places $p \mid D$. The conductor is preserved under the action of $\Gamma$ on $x$.

Let $\mathrm{Opt}(\mathcal{O}_{c^2d}, \mathcal{O}_C) := \{i : k \longrightarrow C \mid i^{-1}(\mathcal{O}_B) = O_{c^2d}\}/\Gamma$ be the set of *optimal embeddings.*

**Lemma 2.9** (Eichler, 9.6)**.** *There is an equality*

$$|\mathrm{Opt}(\mathcal{O}_{c^2d}, \mathcal{O}_C)| = \delta(d, D) h(c^2d).$$

Thus we see

$$\begin{aligned}\sum_{x\in\Gamma\backslash L(t)} |\Gamma_x|^{-1} &= \sum_{c|n,\ (c,D)=1} |\mathrm{Opt}(\mathcal{O}_{c^2d}, \mathcal{O}_C)| \cdot |\mathcal{O}_{c^2d}^{\times}|^{-1} \\ &= \delta(d,D) \sum_{c|n,\ (c,D)=1} h(c^2d)/w(c^2d) \\ &= \delta(d,D) H_0(t,D).\end{aligned}$$

□

## 2.4 Comparison with an Eisenstein series

The point is, that modularity of $\phi_{\mathrm{deg}}$ is not proved by checking the transformation property. Instead, one compares the degrees which were computed above with the Fourier coefficients of an explicit Eisenstein series

$$\mathcal{E}(\tau, s; D), \quad \tau \in \mathfrak{h}, s \in \mathbb{R}_{>0}$$

This series is defined in a canonical way through an adelic formalism in Chapter 6 of [13]. For fixed $s$, this function transforms like a modular form of weight $1+s$. A direct computation (see Chapter 8 of [13]) shows that

$$\mathcal{E}(\tau, \frac{1}{2}; D) = -\frac{1}{12}\prod_{p|D}(p-1) + \sum_{m>0} 2\delta(d;D)H_0(m;D)q^m.$$

This implies the modularity of $\phi_{\mathrm{deg}}$.

# 3 Properness of $\mathcal{M}$

## 3.1 Valuative Criteria for stacks

Recall the following definitions, see [1].

**Definition 3.1.** Let $S$ be noetherian scheme and $f : \mathfrak{X} \longrightarrow S$ Deligne-Mumford stack of finite type.
i) $f$ is *separated* if the diagonal $\mathfrak{X} \longrightarrow \mathfrak{X} \times_S \mathfrak{X}$ is proper.
ii) $f$ is *proper* if it is of finite type, separated and universally closed.

**Remark 3.2.** Universally closed refers to the underlying Zariski spaces $|\mathfrak{X}'| \longrightarrow |S'|$ for each scheme $S'/S$.

Now by definition, the diagonal is representable so the valuative criterion for properness yields the valuative criterion for separatedness.

**Proposition 3.3** (Valuative Criterion for Separatedness)**.** *Let $S$ be a noetherian scheme, $f : \mathfrak{X} \longrightarrow S$ of finite type with separated quasi-compact diagonal $\mathfrak{X} \longrightarrow \mathfrak{X} \times_S \mathfrak{X}$. Then $f$ is separated if and only if:*
*For all DVR $R$ over $S$ and any two morphisms $g_1, g_2 : \mathrm{Spec}\, R \longrightarrow \mathfrak{X}$ over $S$, any isomorphism between the restrictions of $g_1$ and $g_2$ to the generic point lifts to $R$.*

Similarly, we have a valuative criterion of properness. (You can take this as definition if you wish.)

**Proposition 3.4** (Valuative Criterion of Properness). *$S$ noetherian scheme, $f : \mathfrak{X} \longrightarrow S$ of finite type and separated. Then $f$ is proper if and only if:*
*For all DVR $R$ over $S$ with field of fractions $K$ and all $g : \operatorname{Spec} K \longrightarrow \mathfrak{X}$ over $S$, there exists a finite extension $K'/K$ such that $g$ lifts to $R'$, the integral closure of $R$ in $K'$.*

In both propositions, we can restrict to complete $R$ with algebraically closed residue field.

## 3.2 Néron-models and the semi-stable reduction theorem

The reference is [3][Chap. 7].

We would like to apply these criteria to moduli of abelian varieties. Let $R$ be a DVR with field of fractions $K$ and residue field $k$. Let $A_K$ be an abelian variety over $K$.

**Definition 3.5.** A Néron model of $A_K$ over $R$ is a smooth separated finite type scheme $X \longrightarrow \operatorname{Spec} R$ such that $A_K \cong X \otimes_R K$ with the following universal property. For all smooth schemes $T \longrightarrow \operatorname{Spec} R$, every morphism $T \otimes_R K \longrightarrow A_K$ extends uniquely to $T \longrightarrow X$.

**Remark 3.6.** i) A Néron model is unique up to unique isomorphism.
ii) By the universal property, the group structure of $A_K$ lifts to the Néron model $X$.

**Theorem 3.7** (See Chapter 1 of BLR). *i) Any abelian variety $A_K/K$ admits a Néron model $A/R$.*
*ii) An abelian scheme $A/R$ is a Néron model of its generic fibre.*

Note that the Néron model of an abelian variety is not generally an abelian scheme. Indeed, the definition does not include properness.

**Example 3.8.** Let $E_K/K$ be an elliptic curve with Néron model $E/R$. Let $E_k^0$ be the connected component of 0 of the special fiber of $E$. There are three (mutually exclusive) possibilities.
i) $E$ is an elliptic curve over $R$ (good reduction).
ii) $E_k^0$ is a torus (multiplicative/semi-stable reduction).
iii) $E_k^0 \cong \mathbb{G}_a$ (bad/unstable reduction).

Note that formation of the Néron model does not necessarily commute with extending the base field $K'/K$.

**Theorem 3.9** (Grothendieck, Semi-stable Reduction theorem). *Let $R$ be a DVR with field of fractions $K$ and let $A_K/K$ be an abelian variety. Then $A_K$ has* potential semi-stable reduction. *This means, there exists a finite extension $K'/K$ such that the Néron model of $A_K \otimes_K K'$ over $R'$ has a special fiber, which is an extension of an abelian variety by a torus.*

**Example 3.10.** Let $E_K/K$ be an elliptic curve. Then after a suitable base change $K'/K$, the elliptic curve $E_K \otimes_K K'$ will fall into case i) or ii).

## 3.3 Application to $\mathcal{M}$

If $B \cong M_2(\mathbb{Q})$, then $\mathcal{M}$ is the modular curve. We know that it is not proper over $\operatorname{Spec} \mathbb{Z}$. It has to be compactified at the cusp. If $B \ncong M_2(\mathbb{Q})$, we have the following result.

**Proposition 3.11.** *Let $B \neq M_2(\mathbb{Q})$. Then the Shimura curve $\mathcal{M}/\operatorname{Spec} Z$ associated to $B$ is proper.*

*Proof.* We verify the valuative criteria above.
*Separatedness:* The conditions of the criterion are satisfied. The separatedness follows from the Néron property for abelian schemes.

*Properness:* Let $(A_K, \iota_K) \in \mathcal{M}(K)$ and let $K'$ be a finite extension from the semi-stable reduction theorem. Let $A$ be the Néron model over $R'$. The action of $\mathcal{O}_B$ lifts to $A$ by the universal property. Let

$$0 \longrightarrow T \longrightarrow A_s^0 \longrightarrow B \longrightarrow 0$$

be exact with $B$ an abelian variety over the residue field of $R$. We need to show that $\dim T = 0$.

It is a basic fact that $\operatorname{Hom}(T, B) = 0$. ($T$ cannot surject onto an elliptic curve, which would be the image of a non-trivial homomorphism.) So the action of $\mathcal{O}_B$ on $A_s^0$ induces an action of $\mathcal{O}_B$ on $T$. But $\mathcal{O}_B$ cannot map to $M_2(\mathbb{Z})$. (Here the assumption $B \neq M_2(\mathbb{Z})$ enters.) Thus $\dim T = 0$ and $A_K \otimes_K K'$ has good reduction. So the point $(A_K, \iota_K)$ lifts to $R'$. By the valuative criterion, $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$ is proper. □

# 4 Structure of $\mathcal{M}[D(B)^{-1}]$ and $\mathcal{Z}(t)[D(B)^{-1}]$

We have seen that the generic fibre $\mathcal{M}_{\mathbb{C}}$ is smooth. For abstract reasons, this extends to almost all fibers. We will now show that $\mathcal{M}$ is smooth away from the discriminant.

**Proposition 4.1.** *The morphism $\mathcal{M}[D(B)^{-1}] \longrightarrow \operatorname{Spec}\mathbb{Z}[D(B)^{-1}]$ is smooth of relative dimension 1.*

**Corollary 4.2.** *$\mathcal{M}[D(B)^{-1}]$ is a regular 2-dimensional stack. It is flat with geometrically connected fibers over $\mathbb{Z}[D(B)^{-1}]$.*

Note that we cannot prove this fiber by fiber since we do not know flatness of $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$. But we can check the smoothness of the fibre over $p \nmid D(B)$ at each closed point of $\mathcal{M} \otimes \mathbb{F}_p$. So let $\mathbb{F} := \overline{\mathbb{F}_p}$ and $(A, \iota) \in \mathcal{M}(\mathbb{F})$. Let $W := W(\mathbb{F})$ be the ring of Wittvectors of $\mathbb{F}$. It equals the strict completion of $\mathbb{Z}_{(p)}$, $W = \widehat{\mathbb{Z}_p^{ur}}$. Let $\hat{\mathcal{O}}_{\mathcal{M},(A,\iota)}$ be the strict local ring in $(A, \iota)$. Then we need to show the induced morphism on formal schemes

$$\operatorname{Spf}\hat{\mathcal{O}}_{\mathcal{M},(A,\iota)} \longrightarrow \operatorname{Spf} W$$

is formally smooth. The source is of course the formal deformation space of $(A, \iota)$.

The crucial point is that the deformation space of $(A, \iota)$ only depends on the *p-divisible group* of $A$. We will explain this now.

## 4.1 $p$-divisible groups and the theorem of Serre-Tate

A reference is [7].

**Definition 4.3.** Let $S$ be a scheme. A *p-divisible group of height $h$* over $S$ is an inductive system $(X_n)_{n\in\mathbb{N}}$ of finite flat commutative group schemes $X_n/S$ such that

i) $X_n$ has order $p^{nh}$.
ii) For all $n$, the following sequence is exact:

$$0 \longrightarrow X_n \longrightarrow X_{n+1} \xrightarrow{p^n} X_{n+1}.$$

If $X = (X_n)/S$ is a $p$-divisible group (and $S$ connected), we define the dimension $\dim X := \mathrm{rk}_{\mathcal{O}_S} \mathrm{Lie}(X_1)$.

**Example 4.4.** i) $\mathbb{Q}_p/\mathbb{Z}_p := \varinjlim p^{-n}\mathbb{Z}/\mathbb{Z}$ is a $p$-divisible group of height 1 and dimension 0. Here $X_n$ is the constant group scheme $X_n := p^{-n}\mathbb{Z}/\mathbb{Z}$. The transition maps $X_n \longrightarrow X_{n+1}$ are induced from the inclusions $p^{-n}\mathbb{Z} \hookrightarrow p^{-n-1}\mathbb{Z}$.
ii) $\mu_{p^\infty} := \varinjlim \mu_{p^n}$ is of height 1 and dimension 1. Here $X_n$ is the $p^n$-torsion of $\mathbb{G}_m$.
iii) Let $A$ be an abelian variety. Then $A(p) := \varinjlim A[p^n]$ is a $p$-divisible group of height $2\dim A$ and dimension $\dim A$.

For two $p$-divisible groups $X = (X_n)_n$ and $Y = (Y_n)_n$, we have

$$\mathrm{Hom}(X, Y) = \varprojlim \mathrm{Hom}(X_n, Y_n).$$

Note that $\mathbb{Z}$ and hence $\mathbb{Z}_p$ acts naturally on each $p$-divisible group. So $\mathrm{Hom}(X, Y)$ is a $\mathbb{Z}_p$-module. For example, if $S$ is connected and $p$ locally nilpotent in $\mathcal{O}_S$, then $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty}) = \{0\}$ and $\mathrm{End}(\mu_{p^\infty}) = \mathrm{End}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Z}_p$.

If $A, B$ are abelian varieties, then

$$\mathrm{Hom}(A, B) \hookrightarrow \mathrm{Hom}(A, B) \otimes \mathbb{Z}_p \hookrightarrow \mathrm{Hom}(A(p), B(p)).$$

**Theorem 4.5** (Serre-Tate). *Let $A/S$ be an abelian scheme with $p$ locally nilpotent in $\mathcal{O}_S$. Let $S \hookrightarrow S'$ be a nilpotent thickening.*
*i) There is a bijection*

$$\begin{array}{c}\{\textit{Deformations of } A \textit{ to } S'\} = \{\textit{Deformations of } A(p) \textit{ to } S'\} \\ A' \mapsto A'(p)\end{array}.$$

*ii) Let $A'$ and $B'$ be two abelian schemes over $S'$ and let $f : A' \times_{S'} S \longrightarrow B' \times_{S'} S$ be a homomorphism. Then $f$ lifts to a homomorphism $A' \longrightarrow B'$ if and only if $f(p)$ lifts to $A'(p) \longrightarrow B'(p)$.*

**Remark 4.6.** Outside of characteristic $p$, the $p$-divisible group $A(p)$ encodes the same information as the $p$-adic Tate-module $T_p(A)$.

**Example 4.7.** Let $Ell/\operatorname{Spec}\mathbb{Z}$ be the moduli stack of elliptic curves. It is smooth of dimension 1 over $\mathbb{Z}$. This follows immediately from the infinitesimal criterion since there is no obstruction to lifting the coefficients of a Weierstraß equation.

Now let $X := E(p)/\mathbb{F}$ be a $p$-divisible group which appears as the $p$-torsion of an elliptic curve. Then by the Theorem of Serre-Tate, its deformation space over $\operatorname{Spf} W(\mathbb{F})$ is formally smooth of dimension 1, i.e. isomorphic to $\operatorname{Spf} W(\mathbb{F})[[t]]$.

**Definition 4.8.** Let $X$ and $Y$ be two $p$-divisible groups over a (qc) scheme $S$. A *quasi-homomorphism* is an element of

$$\mathrm{Hom}^0(X, Y) := \mathrm{Hom}(X, Y) \otimes \mathbb{Q}.$$

A *quasi-isogeny* is an invertible quasi-homomorphism.

**Theorem 4.9** (Dieudonné-Manin)**.** *Let $k$ be an algebraically closed field in characteristic $p$ and let $(pdiv^0/k)$ be the category of $p$-divisible groups with quasi-homomorphism. Then $(pdiv^0/k)$ is semi-simple and there is a bijection*

$$\begin{aligned} \mathbb{Q} \cap [0,1] &\cong \Sigma(pdiv^0/k) \\ \lambda &\mapsto X_\lambda. \end{aligned}$$

*If $0 \leq r \leq s$ such that $(r,s) = 1$, then $X_{r/s}$ has height $s$ and dimension (of the tangent space) $r$.*

**Definition 4.10.** Let $X/k$ be a $p$-divisible group. We decompose it into simple factors in the isogeny categroy $X \sim X^0 \times \ldots X^n$. Let $X^i$ correspond to the rational number $\lambda_i$ under the Dieudonné-Manin classification. Let us assume that $\lambda_i \leq \lambda_{i+1}$. Then the sequence

$$(\lambda_0, \lambda_1, \ldots, \lambda_n)$$

is called the *slope sequence* or *slopes* of $X$.

**Example 4.11.** The group $\mathbb{Q}_p/\mathbb{Z}_p$ has height 1 and dimension 0, so it has slope $0/1$. The group $\mu_{p^\infty}$ has slope $1/1$.

**Example 4.12.** Let $E/\mathbb{F}$ be an elliptic curve. There are two possibilities for the slopes of $X := E(p)$, which has height 2 and dimension 1. These are $(0/1, 1/1)$ and $(1/2)$.

The first case corresponds to $E$ being ordinary, the second to $X$ being supersingular. Note that this yields a way to define $X_{1/2}$. Also note, that the two groups of height 2 and dimension 1 are even unique up to isomorphism, not only up to isogeny.

To any $p$-divisible group $X = (X_n)/S$ we associate its dual $X^\vee := \varinjlim X_n^\vee$. Here $X_n^\vee$ is the dual in the category of finite commutative group schemes. The transition maps of the inductive systems are the duals of multiplication by $p$

$$p \cdot \mathrm{id}_{X_{n+1}} : X_{n+1} \longrightarrow X_n.$$

If $X/\operatorname{Spec} k$ is simple and $X \leftrightarrow r/s$ under the Dieudonné-Manin theorem, then $X^\vee \leftrightarrow (s-r)/s$.

Furthermore,

$$A(p)^\vee \cong A^\vee(p)$$

are canonically isomorphic, where $A^\vee$ denotes the dual abelian variety.

**Corollary 4.13.** *Let $A/k$ be an abelian variety over an algebraically closed field of characteristic $p$. Let $\lambda_0 \leq \ldots \leq \lambda_n$ be the slopes of $A(p)$ (with multiplicities). Then*

$$\lambda_i = 1 - \lambda_{n-i}.$$

*Proof.* Let $A^\vee/k$ be the dual abelian variety. Since $A$ is projective, there exists an isogeny $A \longrightarrow A^\vee$. Thus their $p$-divisible groups are isogeneous,

$$A(p) \sim A^\vee(p) \cong A(p)^\vee.$$

Now we compare the slopes of $A(p)$ and $A^\vee(p)$. □

**Corollary 4.14.** *Let $A/\mathbb{F}$ be an abelian surface and set $X := A(p)$. Then the only possibilities for the slopes of $A(p)$ are*

$$(0,0,1,1), (0,1/2,1) \textit{ and } (1/2,1/2).$$

## 4.2 Application to $\mathcal{M}$

*Proof of the smoothness.* Let $(A,\iota) \in \mathcal{M}(\mathbb{F})$ where $\mathbb{F} = \mathbb{F}_p^{\mathrm{alg}}$. We want to compute the deformation space of $(A,\iota)$. This is the same as the deformation space of $(A(p), \iota(p))$, where $\iota(p)$ is the induced action of $\mathcal{O}_B \otimes \mathbb{Z}_p$ on $A(p)$.

Now $p \nmid D(B)$ and hence $\mathcal{O}_B \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$. Using the idempotents of $M_2(\mathbb{Z}_p)$, we can write $A(p) = Y^2$ for some $p$-divisible group of height 2. Then deformations of $(A,\iota)$ are in bijection with deformations of $Y$ (without any action!).

We have seen above that any group of height 2 and dimension 1 embeds into an elliptic curve. Thus $Def(Y)$ is formally smooth of relative dimension 1 over $W(\mathbb{F})$. □

**Remark 4.15.** In general, if $X/\mathbb{F}$ is a $p$-divisible group of height $h$ and dimension $d$, then its deformation space is formally smooth of relative dimension $h(h-d)$ over $W(\mathbb{F})$. This follows from Theorem 4.18 below.

## 4.3 Deformation theory of $p$-divisible groups

We collect some important results. In this subsection $p$ is some prime and $S$ is a (qc) scheme with $p$ locally nilpotent in $\mathcal{O}_S$.

**Proposition 4.16** (Unramifiedness of Hom). *Let $X$ and $Y$ be $p$-divisible groups over $S$ and let $f : X \longrightarrow Y$ be a quasi-homomorphism. Then there exists a closed subscheme $Z(f) \subset S$ such that $u : T \longrightarrow S$ factors over $Z(f)$ if and only if $u^*f$ is a homomorphism $u^*f : u^*X \longrightarrow u^*Y$.*

**Proposition 4.17** (Drinfeld, Rigidity). *Let $S \hookrightarrow S'$ be a nilpotent thickening. Let $X$ and $Y$ be two $p$-divisible groups over $S'$. Then the reduction of homomorphisms defines a bijection of* quasi-*homomorphisms*

$$\mathrm{Hom}^0(X,Y) \xrightarrow{=} \mathrm{Hom}^0(X \times_{S'} S, Y \times_{S'} S).$$

Injectivity follows from the unramifiedness of Hom. Surjectivity can be formulated as follows. Let $f : X \times_{S'} S \longrightarrow Y \times_{S'} S$ be a homomorphism. Then there exists (locally on $S$) an integer $N$ such that $p^N f$ extends to $S'$.

**Theorem 4.18** (Grothendieck, Crystalline Deformation Theorem). *Let $S \hookrightarrow S'$ be a square-zero thickening, i.e. $S = V(\mathcal{I})$ for some ideal $\mathcal{I} \subset \mathcal{O}_{S'}$ with $\mathcal{I}^2 = 0$. (More generally, $S \subset S'$ is a PD-thickening.) Then there exists a functor*

$$\begin{aligned} \mathbb{D}_{S'} : (pdiv/S) &\longrightarrow (\mathcal{O}_{S'}\text{-modules} + \text{filtration}) \\ X &\mapsto (\mathbb{D}_{S'}(X),\ \mathcal{F} \subset \mathbb{D}_{S'}(X) \otimes_{\mathcal{O}_{S'}} \mathcal{O}_S) \end{aligned}$$

*with the following properties. The $\mathcal{O}_{S'}$-module $\mathbb{D}_{S'}(X)$ is locally free of rank equal to the height of $X$. And $\mathcal{F} \subset \mathbb{D}_{S'}(X) \otimes \mathcal{O}_S$ is locally a direct summand, locally free of rank $\dim(X^\vee) = (\mathrm{ht}(X) - \dim(X))$. Furthermore, there is a bijection*

$$\{\textit{Lifts } \mathcal{F}' \textit{ of } \mathcal{F} \textit{ to locally direct summands of } \mathbb{D}_{S'}(X)\} \cong \{\textit{Deformations of } X \textit{ to } S'\}.$$

*If $X'/S'$ is the deformation corresponding to $\mathcal{F}'$ and if $f \in \mathrm{End}(X)$, then $f$ lifts to $X'$ if and only if $\mathbb{D}(f)\mathcal{F}' \subset \mathcal{F}'$.*

Terminology: $\mathbb{D}_{S'}(X)$ is called the *crystal of $X$ evaluated at $S'$*. The submodule $\mathcal{F}$ is called the *Hodge filtration.*

## 4.4 Application to $\mathcal{Z}(t)$

**Proposition 4.19.** *The* 0*-cycle* $\mathcal{Z}(t)_{\mathbb{C}}$ *is nonempty if and only if the imaginary quadratic field* $\mathbb{Q}(\sqrt{t})$ *embeds into* $B$*. In this case, the stack* $\mathcal{Z}(t)$ *is flat over* $\operatorname{Spec}\mathbb{Z}[D(B)^{-1}]$ *and represents a relative divisor.*

*Proof.* The first assertion was explained in the first chapter. The second assertion can be proved after completion.

Let $p \nmid D(B)$, $\mathbb{F} = \overline{\mathbb{F}_p}$ and $W = W(\mathbb{F})$ as usual. Let $(A, \iota, x) \in \mathcal{Z}(t)(\mathbb{F})$. It is enough to show that $Def(A, \iota, x) \longrightarrow Def(A, \iota)$ is a relative divisor over $\operatorname{Spf} W$.

*Step 1: Reduction to p-divisible groups.* Let $X := A(p)$ be the $p$-divisible group of $A$. It has height 4 and dimension 2. It is endowed with the induced action $\iota : \mathcal{O}_B \longrightarrow \operatorname{End}(X)$ and the induced endomorphism $x \in \operatorname{End}_{\mathcal{O}_B}(X)$. The action $\mathcal{O}_B \circlearrowleft X$ extends to an action of $\mathcal{O}_B \otimes \mathbb{Z}_p$. By Serre-Tate, $Def(A, \iota, x) = Def(X, \iota, x)$ and $Def(A, \iota) = Def(X, \iota)$.

Now $p \nmid D(B)$, so $\mathcal{O}_B \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$. Choosing two complementary idempotents yields a decomposition $(X, \iota, x) = (Y^2, \mathrm{can}, x)$. Here $Y$ is some $p$-divisible group of height 2 and dimension 1 and can is the canonical action of $M_2(\mathbb{Z}_p)$ on $Y^2$. The endomorphism $x$ is $\mathcal{O}_B$-linear and so is diagonal,

$$x = \begin{pmatrix} y & \\ & y \end{pmatrix}.$$

Then $Def(X, \iota, x) \subset Def(X, \iota)$ equals $Def(Y, y) \subset Def(Y)$.

*Claim: Let* $Y$ *be a p-divisible group of height* 2 *and dimension* 1*. Let* $y \in \operatorname{End}(Y) \setminus \mathbb{Z}_p$*. Then* $Def(Y, y) \subset Def(Y)$ *is a relative divisor (over* $W(\mathbb{F})$*).*

*Step 2:* $Def(Y, y) \subset Def(Y)$ *is a closed formal subscheme.* Proposition 4.17 yields a unique way to extend $y$ to a *quasi*-endomorphism on $Def(Y)$. Then Proposition 4.16 yields the result.

*Step 3:* $Def(Y, y) \subset Def(Y)$ *is generated by one equation.* Let $S'/W$ be artinian and consider $\mathcal{Y}' \in Def(Y)(S')$ a deformation of $Y$. Let $S \hookrightarrow S'$ be a square-zero closed subscheme such that $y$ lifts to $\mathcal{Y}' \times_{S'} S$. Let

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathbb{D}_{S'}(\mathcal{Y}') \longrightarrow \mathcal{Q}' \longrightarrow 0$$

be the Hodge filtration of $\mathcal{Y}'$, as in Theorem 4.18. Let $\mathbb{D}(y)$ be the endomorphism of the Dieudonné crystal $\mathbb{D}(\mathcal{Y}')$ induced by $y$. Then $\mathbb{D}(y)\mathcal{F} \subset \mathcal{F}$, i.e. $y$ preserves the Hodge filtration of $\mathcal{Y}' \times_{S'} S$.

By 4.18, $y$ lifts to $\mathcal{Y}$ if and only if $\mathbb{D}(y)\mathcal{F}' \subset \mathcal{F}'$. Equivalently, if the composition

$$\mathcal{F}' \longrightarrow \mathbb{D}_{S'}(\mathcal{Y}') \xrightarrow{\mathbb{D}(y)} \mathbb{D}_{S'}(\mathcal{Y}') \longrightarrow \mathcal{Q}'$$

is zero. But this is a homomorphism of "line bundles" (on the zero-dimensional scheme $S'$), so its vanishing locus is obviously described by one equation. Note that we did not use $y \notin \mathbb{Z}_p$. If this is the case, then the defining equation is just 0.

*Step 4:* $Def(Y, y) \subset Def(Y)$ *is a relative divisor.*
The deformation space $Def(Y) \cong \operatorname{Spf} W(\mathbb{F})[[v]]$ is regular, so it is enough to show that $Def(Y, y) \neq Def(Y)$ to prove that $Def(Y, y)$ is a divisor. This divisor is relative if and only if $V(p) = \operatorname{Spf} \mathbb{F}[[v]] \nsubseteq Def(Y, y)$. The claim now follows from the results in [2, Chap. 8 and 9].

Note that if we are only interested in a special endomorphism $y$ induced from a point $(A, \iota, x) \in \mathcal{Z}(t)(\mathbb{F})$, then it is enough to understand the supersingular case, i.e. [2, Chap. 8]. Namely, $Def(Y, y)$ is then the formal deformation space of $(A, \iota, x)$ and it is enough to show that

$$\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p \not\subseteq \mathrm{Image}(\mathcal{Z}(t) \longrightarrow \mathcal{M}).$$

But this can be checked at a supersingular point in $\mathcal{M}(\mathbb{F})$. □

# 5 Structure of $\mathcal{M}$ at $p \mid D(B)$ ($p$-adic uniformization)

The reference for this section is [6]. Everything can be found there.

The complex fiber $\mathcal{M}_{\mathbb{C}}^{\mathrm{an}}$ has a uniformization by the upper half-plane. Similarly if $p \mid D(B)$, then the $p$-adic completion $\mathcal{M}_p$ of $\mathcal{M}$ along $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{F}_p$ has a uniformization by the *Drinfeld upper half-plane* $\Omega$.

We will now explain this $p$-adic uniformization. To do this, we also have to define $\Omega$ and to explain the concept of a *Rapoport-Zink space.*

## 5.1 The theorem of Honda and Tate

Recall that an *isogeny* $A \longrightarrow B$ of abelian varieties is a surjective homomorphism with finite kernel. A *quasi-isogeny* is an invertible element in $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes \mathbb{Q}$. Recall that the category of abelian varieties *up to quasi-isogeny* over a fixed field is semi-simple.

Fix $q$ and denote by $\sum AV^0(\mathbb{F}_q)$ the isomorphism classes of simple objects of the category of abelian varieties up to isogeny. Here, simple and isogenies are as abelian varieties over $\mathbb{F}_q$.

Let

$$W(q) := \{\alpha \in \overline{\mathbb{Q}} \mid \forall \rho : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C},\ |\rho(\alpha)| = q^{1/2}\}$$

be the set of Weil-$q$-numbers of weight 1. The absolute Galois group $G_{\mathbb{Q}}$ acts on it.

If $A \in \sum AV^0(\mathbb{F}_q)$, then $\mathrm{End}^0(A)$ is a division algebra over $\mathbb{Q}$ and the $q$-Frobenius $\pi_A$ generates a field extension $\mathbb{Q}(\pi_A)/\mathbb{Q}$. The Riemann hypothesis for varieties over $\mathbb{F}_q$ states that $\pi_A$ is a Weil-$q$-number, $\pi_A \in G_{\mathbb{Q}} \backslash W(q)$.

**Theorem 5.1** (Honda-Tate)**.** *The map $A \mapsto \pi_A$ induces a bijection between $\sum AV^0(\mathbb{F}_q)$ and $G_{\mathbb{Q}} \backslash W(q)$. Furthermore if $A$ corresponds to $\pi_A$, then $D_A := \mathrm{End}^0(A)$ is a division algebra with center $\mathbb{Q}(\pi_A)$. One knows that*

$$2 \dim A = [\mathbb{Q}(\pi_A) : \mathbb{Q}][D_A : \mathbb{Q}(\pi_A)]^{1/2}.$$

*This division algebra is unramified outside $p$.*

**Example 5.2.** Let $E/\mathbb{F}_q$ be a supersingular elliptic curve. Then $\pi_E$ is of degree $q$ and satisfies $\pi_E^2 = q$. (A purely inseparable morphism between curves factors over the Frobenius.) It follows that $\pi_E = \sqrt{q}$.

**Lemma 5.3.** *Let $p \mid D(B)$ be a prime and set $\mathbb{F} := \mathbb{F}_p^{\mathrm{alg}}$. Then the points in $\mathcal{M}(\mathbb{F})$ form a single isogeny class (wrt. $\mathcal{O}_B$-linear isogeny.)*

*Proof.* We first show that for every $(A, \iota) \in \mathcal{M}(\mathbb{F})$, $A$ is isogeneous to a product of supersingular elliptic curves.

First note that $A(p)$ has an action of $\mathcal{O}_B \otimes \mathbb{Z}_p$ which is a maximal order in a quaternion algebra over $\mathbb{Q}_p$. The $p$-divisible group $A(p)$ is of dimension 2 and height 4. We decompose it into isotypical components in the isogeny category. $B \otimes \mathbb{Q}_p$ acts on each isotypic component.

The slopes 0 and 1 cannot occur since $B \otimes \mathbb{Q}_p$ can act on no smaller power than $(\mathbb{Q}_p/\mathbb{Z}_p)^4$ or $\mu_{p^\infty}^4$. So by Corollary 4.14, the slopes are $(1/2, 1/2)$ and $A$ is supersingular. Now assume that $A$ is defined over $\mathbb{F}_q$. It follows from the structure of $A(p)$ that $\pi_A^2/q$ is integral such that $|\rho(\pi_A^2/q)| = 1$ for all embeddings $\rho : \mathbb{Q}(\pi_A) \longrightarrow \mathbb{C}$. It is thus a root of unity. By extending scalars, we can assume $\pi_A = \sqrt{q} \in \mathbb{Q}$.

But this Weil number corresponds to a supersingular elliptic curve under the Honda-Tate bijection. It follows that $A$ cannot be simple. It is thus isogeneous to a product of two supersingular elliptic curves, $A \sim E \times E$.

Via this isogeny, we transport the $\mathcal{O}_B$-action to $E \times E$. This is only an action by quasi-isogenies! But up to isogeny, there is a unique action $B \longrightarrow \mathrm{End}^0(E \times E)$. Namely recall that $\mathrm{End}^0(E \times E) \cong D^{(p)}/\mathbb{Q}$, the quaternion algebra with invariant 1/2 precisely at $p$ and $\infty$. By Skolem-Noether, all embeddings $B \longrightarrow M_2(D^{(p)})$ are conjugate. This finishes the proof. □

From here on, we fix a base point $(A_0, \iota_0) \in \mathcal{M}(\mathbb{F})$. We have seen that all points in $\mathcal{M}(\mathbb{F})$ are isogeneous to $(A_0, \iota_0)$. So we can *overparametrize* $\mathcal{M}(\mathbb{F})$ by parametrizing all $\mathcal{O}_B$-linear quasi-isogenies $(A_0, \iota_0) \longrightarrow (A, \iota)$. We now motivate the characteristic $p$ theory with the theory over $\mathbb{C}$.

## 5.2 Isogenies over $\mathbb{C}$

Let $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ be the integral adeles and $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$. There is an exact sequence

$$0 \longrightarrow \widehat{\mathbb{Z}} \longrightarrow \mathbb{A}_f \longrightarrow \mathbb{A}_f/\widehat{\mathbb{Z}} \longrightarrow 0.$$

Let $A/\mathbb{C}$ be an abelian variety of dimension $d$. Define the *Tate module* $T(A) := \varprojlim A[n]$ and the *rational Tate module* $\mathbb{V}(A) := T(A) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}_f$. Tensoring the above sequence with $T(A)$ yields

$$0 \longrightarrow T(A) \longrightarrow \mathbb{V}(A) \xrightarrow{\alpha} \varinjlim A[n] \longrightarrow 0.$$

Giving an isogeny $f : A \longrightarrow B$ is the same as giving its kernel, a finite flat group scheme $K \subset A$. If $f$ has degree $n$, then $K \subset A[n]$. Then $\alpha^{-1}(T(B))$ is a superlattice $T(A) \subset \Lambda \subset \mathbb{V}(A)$. This yields a bijection

$$\{\text{Quasi-isogenies } f : A \dashrightarrow B\} = \{\text{lattices } \Lambda \subset \mathbb{V}(A)\} = GL(\mathbb{V}(A))/GL(T(A)).$$

Now consider two quasi-isogenies $f : A \longrightarrow B$ and $f' : A \longrightarrow B'$ corresponding to lattices $\Lambda, \Lambda' \subset \mathbb{V}(A)$. Then $B \cong B'$ if and only if there exists a quasi-isogeny $g : B \longrightarrow B'$ such that $gT(B) = T(B')$ in $\mathbb{V}(B')$. If $\varphi : B \cong B'$, then $\psi := (f')^{-1}\varphi f \in I(A)$, the group of quasi-isogenies of $A$, and $\psi\Lambda = \Lambda'$. Choosing a $\widehat{\mathbb{Z}}$-basis of $T(A)$, we arrive at:

**Remark 5.4** (Adelic description of an isogeny class over $\mathbb{C}$)**.** There is a bijection

$$\{\text{Abelian varieties } B, \text{isogeneous to } A\} = I(A)\backslash GL_{2d}(\mathbb{A}_f)/GL_{2d}(\widehat{\mathbb{Z}}).$$

## 5.3 Isogenies in characteristic $p$ and Rapoport-Zink spaces

The reference is [16].

Now if $A$ is in char $p$, the adelic description only works for prime-to-$p$ isogenies. But at $p$, we have to replace the Tate-module by the $p$-divisible group $A(p)$. Oviously, there is a bijection

$$\{\text{finite flat p-grps } K \subset A\} \cong \{\text{subgrps} K \subset A(p)\}.$$

Let $\hat{\mathbb{Z}}^p$, $\mathbb{A}_f^p$, $T^p(A)$ and $\mathbb{V}^p(A)$ be the corresponding rings/modules away from $p$. Then

$$\{\text{quasi-isogenies} A \dashrightarrow B\} = \{\widehat{\mathbb{Z}}^p - \text{lattices} \Lambda \subset \mathbb{V}^p(A) + \text{a quasi-isogeny} A(p) \dashrightarrow B(p)\}. \tag{5.1}$$

This motivates the next definition. Let $W := W(\mathbb{F})$ be the ring of Witt vectors for $\mathbb{F}$. Denote by $Nilp$ the category of schemes over $\operatorname{Spf} W$, i.e. the category of schemes $S/\operatorname{Spec} W$ with $p$ locally nilpotent in $\mathcal{O}_S$. For $S \in Nilp$, we write $\overline{S} := S \otimes_W \mathbb{F}$ for the special fiber.

**Definition 5.5.** Let $\mathbb{X} \longrightarrow \operatorname{Spec} \mathbb{F}$ be $p$-divisible group. On $Nilp$, we define the functor

$$\mathcal{N}_{\mathbb{X}} : Nilp^{opp} \longrightarrow (Set), S \mapsto \{(X, \rho) \mid X/S + \rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \dashrightarrow X \times_S \overline{S}\}/\cong .$$

Two pairs $(X, \rho), (X', \rho')$ are isomorphic if there exists an isomorphism $\gamma : X \longrightarrow X'$ such that $\gamma \circ \rho = \rho'$.

Due to the Lemma of Drinfeld 4.17, objects $(X, \rho)$ as in Definition 5.5 have no automorphisms.

**Theorem 5.6** (Rapoport-Zink)**.** *The functor* $\mathcal{N}_{\mathbb{X}}$ *is representable by a formal scheme, locally formally of finite type over* $\operatorname{Spf} W(\mathbb{F})$.

**Definition 5.7.** Let $\rho : X \longrightarrow Y$ be a quasi-isogeny of $p$-divisible groups over a connected scheme $S$. Then locally on $S$, there exists $N \in \mathbb{N}$ such that $p^N \rho$ is an isogeny. We define the height of $\rho$ by the relation

$$|\ker(p^N \rho)| = p^{\operatorname{ht}(X) \cdot N + \operatorname{ht}(\rho)}.$$

The height is locally constant.

**Remark 5.8.** i) $\mathcal{N}_{\mathbb{X}} = \sqcup_{i \in \mathbb{Z}} \mathcal{N}_{\mathbb{X},i}$ where $\mathcal{N}_{\mathbb{X},i}$ is the locus where $\rho$ has height $i$.
ii) The group of quasi-isogenies $g : \mathbb{X} \longrightarrow \mathbb{X}$ acts on $\mathcal{N}$ via $(X, \rho) \mapsto (X, \rho \circ g^{-1})$.
iii) $\mathcal{N}_{\mathbb{X}}$ only depends on $\mathbb{X}$ up to isogeny.

**Remark 5.9** (Important Remark)**.** Let $(X, \rho) \in \mathcal{N}(S)$ and let $s$ be a quasi-endomorphism of $\mathbb{X}$. Then $\rho^{-1} \circ s \circ \rho$ is a quasi-endomorphism of $X \times_S \overline{S}$. By Drinfeld rigidity 4.17, we can view it as quasi-endomorphism of $X$ itself.

By the unramifiedness of Hom 4.16, there exists a closed formal subscheme $Z(s) \subset \mathcal{N}_{\mathbb{X}}$ such that $(X, \rho) \in \mathcal{N}_{\mathbb{X},s}$ if and only if $\rho s \rho^{-1}$ is an endomorphism of $X$.

## 5.4 The Drinfeld moduli problem

A good reference is [11].

We let $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with maximal order $\mathcal{O}_{B_p} := \mathcal{O}_B \otimes \mathbb{Z}_p$. Our chosen base point $(A_0, \iota_0)$ yields a $p$-divisible group $\mathbb{X} := A_0(p)$ together with an action $\iota_{\mathbb{X}} : \mathcal{O}_{B_p} \longrightarrow \mathrm{End}(\mathbb{X})$. The group $\mathbb{X}$ has height 4 and satisfies the *special condition*, i.e.

$$\forall b \in \mathcal{O}_{B_p} \quad \mathrm{charpol}(\iota(b) \mid \mathrm{Lie}(\mathbb{X}))(T) = \mathrm{charrd}(b)(T).$$

**Remark 5.10.** The pair $(\mathbb{X}, \iota_{\mathbb{X}})/\mathbb{F}$ is unique up to $\mathcal{O}_{B_p}$-linear isogeny.

**Definition 5.11.** We define the *Drinfeld moduli problem* to be the following functor on $Nilp$:

$$\mathcal{N} : S \mapsto \{(X, \iota, \rho) \mid \ldots\}/\cong,$$

where $X/S$ is a $p$-divisible group of height 4, $\iota : \mathcal{O}_{B_p} \longrightarrow \mathrm{End}(X)$ an action of $\mathcal{O}_{B_p}$ such that the special condition is satisfied and $\rho$ a quasi-isogeny

$$\rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \longrightarrow X \times_S \overline{S}.$$

**Lemma 5.12.** *The functor $\mathcal{N}$ is representable by a formal scheme, locally formally of finite type over* $\mathrm{Spf}\, W$.

*Proof.* By Rapoport-Zink, the space of quasi-isogenies $\mathcal{N}_{\mathbb{X}}$ is representable. The locus where the action of $\mathcal{O}_{B_p}$ lifts is a closed subfunctor and the special condition is also closed. □

Let $B^{(p)}/\mathbb{Q}$ be the quaternion algebra with the same invariants as $B$ except at the places $p$ and $\infty$. In other words, $[B^{(p)}]+[B] = [D^{(p)}]$ in the Brauer group of $\mathbb{Q}$. Let $I := B^{(p)}/\mathbb{Q}$ be the algebraic group of units of $B^{(p)}$.

Then $I(\mathbb{Q}_p)$ be the group of automorphisms of $(\mathbb{X}, \iota_{\mathbb{X}})$, i.e. the group of $\mathcal{O}_{B_p}$-linear quasi-isogenies $\mathbb{X} \longrightarrow \mathbb{X}$. There is an isomorphism $I(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$. As in Remark 5.8, $I(\mathbb{Q}_p)$ operates on $\mathcal{N}$ by composition in the framing

$$g \cdot (X, \rho) := (X, \rho \circ g^{-1}).$$

There is a decomposition $\mathcal{N} = \coprod_i \mathcal{N}_i$, where $\mathcal{N}_i$ is characterized by the fact that $\mathrm{ht}(\rho|_{\mathcal{N}_i}) = 2i$. Note that $\mathrm{ht}(\rho \circ g^{-1}) = \mathrm{ht}(\rho) - 2v_p(\det g)$. So $I(\mathbb{Q}_p)$ acts transitively on the $\mathcal{N}_i$.

**Remark 5.13.** The space $\mathcal{N}$ plays the same role in the $p$-adic uniformization as the factor $GL_{2d}(\mathbb{Q}_p)/GL_{2d}(\mathbb{Z}_p)$.

## 5.5 The Drinfeld half-plane

The structure of $\mathcal{N}_0$ was determined by Drinfeld. It is isomorphic to the *Drinfeld half-plane.*

Let $\mathcal{T} = \mathcal{T}(PGL_2(\mathbb{Q}_p))$ be the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$. Its vertices are the homothety classes of lattices in $\mathbb{Q}_p^2$. Two vertices $[\Lambda_0]$ and $[\Lambda_1]$ are joined by an edge if there exist representatives $\Lambda_0$ and $\Lambda_1$ such that $\Lambda_0 \subset \Lambda_1$ of index 1.

For $[\Lambda] \in \mathcal{T}$ a vertex, we define

$$\hat{\Omega}_{[\Lambda]} := (\mathbb{P}(\Lambda))^\wedge \setminus \mathbb{P}(\Lambda)(\mathbb{F}_p).$$

Here $\wedge$ denotes the $p$-adic completion of a scheme. The choice of a basis $\Lambda = (e_0, e_1)$ defines an isomorphism

$$\hat{\Omega}_{[\Lambda]} \cong \operatorname{Spf} \mathbb{Z}_p[T, (T^p - T)^{-1}]^\wedge.$$

For $\Delta = [\Lambda_0, \Lambda_1] \subset \mathcal{T}$ an edge, we define a chart $\hat{\Omega}_\Delta$. Let us do this in coordinates. Apply the elementary divisor theorem to get representatives and bases $\Lambda_0 = (e_0, e_1), \Lambda_1 = (pe_0, e_1)$. Then define

$$\hat{\Omega}_\Delta = \operatorname{Spf} \mathbb{Z}_p[T_0, T_1, (1 - T_0^{p-1}), (1 - T_1^{p-1})^{-1}]^\wedge / T_0 T_1 - p.$$

This chart canonically contains $\hat{\Omega}_{\Lambda_0}$ and $\hat{\Omega}_{\Lambda_1}$.

**Definition 5.14.** The *formal Drinfeld half-plane* $\hat{\Omega}$ is the formal scheme which is defined by gluing $\{\hat{\Omega}_\Delta \mid \Delta \subset \mathcal{T} \text{ simplex}\}$ such that

$$\hat{\Omega}_\Delta \cap \hat{\Omega}_{\Delta'} = \begin{cases} \hat{\Omega}_{\Delta \cap \Delta'} & \text{if } \Delta \cap \Delta' \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $\hat{\Omega}_\Delta \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a configuration of $\mathbb{P}^1$'s, intersecting transversally with dual graph equal to $\mathcal{T}$. Also note that $\hat{\Omega}$ is regular.

Note that $PGL_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ via $g[\Lambda] = [g\Lambda]$. And $g$ defines an isomorphism

$$\mathbb{P}(\Lambda) \xrightarrow{\cong} \mathbb{P}(g\Lambda).$$

This defines an action of $PGL_2(\mathbb{Q}_p)$ on $\hat{\Omega}$. This action is very explicit! The stabilizer of an irreducible component $\mathbb{P}(\Lambda)$ equals $PGL(\Lambda)$. Fix a basis $\Lambda = (e_0, e_1)$ and let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(\Lambda).$$

The $g$ acts as Möbius transformation

$$g : t = e_0/e_1 \mapsto \frac{at + b}{ct + d}.$$

Let $\hat{\Omega}^\bullet := \mathbb{Z} \times \hat{\Omega}$ and endow it with an action of $GL_2(\mathbb{Q}_p)$ by the rule $g : [i, \mathbb{P}(\Lambda)] \xrightarrow{\cong} [i - v_p(\det g), \mathbb{P}(g\Lambda)]$.

**Theorem 5.15.** *Fix an isomorphism $I(\mathbb{Q}_p) \cong GL_2(\mathbb{Q})$. Then there is a G-equivariant isomorphism*

$$\mathcal{N} \cong \hat{\Omega}^\bullet_W := \hat{\Omega}^\bullet \times_{\mathbb{Z}_p} \operatorname{Spf} W.$$

**Corollary 5.16.** *$\mathcal{N}$ is regular of dimension 2 with semi-stable reduction over* $\operatorname{Spf} W$.

Here semi-stable is meant in the following sense: The special fiber $\mathcal{N} \otimes_W \mathbb{F}$ is a configuration of $\mathbb{P}^1$'s which intersect in ordinary double points and each $\mathbb{P}^1$ intersects at least three other $\mathbb{P}^1$'s.

### 5.6 $p$-adic Uniformization of $\mathcal{M}$

We have seen that $(A_0, \iota_0)$ is isogeneous to a product of supersingular elliptic curves. Thus $\mathrm{End}^0(A_0) \cong M_2(D^{(p)})$ and by the centralizer theorem, $\mathrm{End}^0_{\mathcal{O}_B}(A_0) \cong B^{(p)}$.

Recall that we defined $I = B^{(p),\times}$ as algebraic group over $\mathbb{Q}$. Then $I(\mathbb{Q})$ is the group of $\mathcal{O}_B$-linear quasi-isogenies of $A_0$. Furthermore, $I(\mathbb{A}_f^p) = \mathrm{Aut}_B(\mathbb{V}^p(A_0))$ and we let $I(\mathbb{A}_f^p) \supset K^p := \mathrm{Stab}(T^p(A_0))$. This is a maximal compact subgroup. If wanted, we could also identify $B^{(p)} \otimes \mathbb{A}_f^p \cong B \otimes \mathbb{A}_f^p$ such that $K^p = (\mathcal{O}_B \otimes \hat{\mathbb{Z}}^p)^\times$.

Now consider the space $\hat{\Omega}^\bullet_W \times I(\mathbb{A}_f^p)$. To any $S$-valued point $[(X, \iota, \rho), g]$ we associate an abelian surface with $\mathcal{O}_B$-action oves $S$ as follows. First consider $A_0 \times_{\mathbb{F}} \overline{S}$. The datum of $\rho$ and $g$ define an isogeny $\alpha : A_0 \times \overline{S} \longrightarrow \overline{A}$ as in (5.1). Then $b \mapsto \alpha\iota(b)\alpha^{-1} \in \mathrm{End}^0(\overline{A})$ *defines an action* of $\mathcal{O}_B$ on $\overline{A}$. (I.e. these are not only quasi-endomorphisms.) To check this, we have to check that for all $b \in \mathcal{O}_B$, $\alpha\iota(b)\alpha^{-1}$ is an endomorphism of $\overline{A}(p)$ and preserves the Tate module $T^p(\overline{A})$. But these conditions follow from the $\mathcal{O}_B$-linearity of $\rho$ and $g$.

Now $X$ is a deformation of $X \times_S \overline{S}$ to which the action of $\mathcal{O}_B$ lifts. By the theorem of Serre, there is a corresponding deformation $A$ of $\overline{A}$ to which the action of $\mathcal{O}_B$ lifts. Thus we defined a transformation of functors

$$\hat{\Omega}^\bullet_W \times I(\mathbb{A}_f^p)/K^p \longrightarrow \mathcal{M}|_{Nilp_W} \tag{5.2}$$

$$[(X, \iota, \rho), g] \mapsto [\alpha : A_0 \times \overline{S} \longrightarrow \overline{A}, \iota] \mapsto (A, \iota). \tag{5.3}$$

Note that the restriction $\mathcal{M}|_{Nilp_W}$ is nothing but the $p$-adic completion $\mathcal{M}_p = \varinjlim \mathcal{M} \otimes W/p^n$. There is an obvious diagonal action of $I(\mathbb{Q})$ on the left hand side of (5.2) and the morphism factors over the quotient. So we almost proved the following theorem.

**Theorem 5.17** (Cherednik-Drinfeld). *There is an isomorphism of formal stacks*

$$[I(\mathbb{Q})\backslash \hat{\Omega}^\bullet_W \times I(\mathbb{A}_f^p)/K^p] \cong \mathcal{M}_p.$$

**Corollary 5.18.** *The Shimura curve $\mathcal{M}$ is a regular surface. The geometric fibres of $\mathcal{M} \longrightarrow \mathbb{Z}$ are semi-stable and connected.*

Note that the irreducible components of $\hat{\Omega}^\bullet_W$ are in bijection with $GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p) = I(\mathbb{Q}_p)/K_p$.

**Corollary 5.19.** *There is a bijection*

$$I(\mathbb{Q})\backslash I(\mathbb{A}_f)/K^pK_p \xrightarrow{\cong} \{\textit{irred comp of } \mathcal{M}_p\}.$$

This set can be identified with the class group of $B^{(p)}$.

## 6 Structure of Special Cycles at $p \mid D(B)$

Still, the reference is [11].

As in the complex situation, we also uniformize the ($p$-adic completions) $\mathcal{Z}(t)_p$.

## 6.1 Special Cycles on $\widehat{\Omega}^{\bullet}$

Recall that $\mathbb{X} = A_0(p)$ with its action by $\mathcal{O}_{B_p} = \mathcal{O}_B \otimes \mathbb{Z}_p$. Let $V(\mathbb{X}, \iota_{\mathbb{X}}) := \{j \in \mathrm{End}^0_D(\mathbb{X}) \mid \mathrm{tr}(j) = 0\}$ denote the *special endomorphisms* of $\mathbb{X}$. Again we have the $\mathbb{Q}_p$-valued quadratic form $Q(j) = -j^2$. Recall that $\mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}}) \cong M_2(\mathbb{Q}_p)$, so $V(\mathbb{X}, \iota_{\mathbb{X}})$ is isomorphic to the $2 \times 2$ trace 0 matrices over $\mathbb{Q}_p$. There is an inclusion $V(A_0, \iota_0) \subset V(\mathbb{X}, \iota_{\mathbb{X}})$.

**Definition 6.1.** For $0 \neq j \in \mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}})$, we define

$$Z(j) \hookrightarrow \mathcal{N}$$

to be the locus on which $\rho j \rho^{-1}$ is an isogeny. If $j$ is a special endomorphism, then we call this a *special cycle*.

**Remark 6.2.** i) Obviously, the cycle $Z(j)$ only depends on the $\mathbb{Z}_p$-algebra generated by $j$ in $\mathrm{End}^0(\mathbb{X})$.
ii) The action of as element $g \in I(\mathbb{Q}_p)$ on $\mathcal{N}$ defines an isomorphism $g : Z(j) \longrightarrow Z(gjg^{-1})$.

For $g \in I(\mathbb{Q}_p)$, we denote by $\mathcal{N}^g$ the fixed point scheme of $g$. It is empty, if $\det(g) \notin \mathbb{Z}_p^{\times}$.

**Theorem 6.3.** *Let $j \in V(\mathbb{X})$ with $0 \neq Q(j) = \varepsilon p^{\alpha}$ where $\varepsilon \in \mathbb{Z}_p^{\times}$. Then*

$$Z(j) \cong \begin{cases} \emptyset & \text{if } \alpha < 0 \\ \mathcal{N}^j & \text{if } \alpha = 0 \\ \mathcal{N}^{1+j} & \text{if } \alpha > 0. \end{cases}$$

*Proof.* If $\alpha < 0$, then $j$ has negative height. Thus it cannot be an endomorphism of any $X$.
If $\alpha = 0$ and if $\rho j \rho^{-1}$ is an endomorphism of $X$, then it is even an automorphism of $X$. This is the case if and only if it defines an isomorphism $(X, \iota, \rho) \cong (X, \iota, \rho j)$ which is equivalent to $X \in \mathcal{N}^j$.
If $\alpha > 0$, then we use the equality $Z(j) = Z(1 + j)$. Now the previous argument did not need $\mathrm{tr}(j) = 0$ and so also holds for $1 + j$. □

## 6.2 Structure of $Z(j)$

Note the picture on page 9 in [11].

Now fix $j$ with $Q(j) \neq 0$ and $\alpha \geq 0$. Let $\mathcal{N}^{ord} = \hat{\Omega}^{\bullet;ord}_W$ be the ordinary locus, i.e. the complement of the intersection points of the $\mathbb{P}^1$'s. Let $Z(j)^{ord} = Z(j) \cap \mathcal{N}^{ord}$.

In general, $Z(j)$ is purely 1-dimensional. But it is not a divisor (i.e. locally generated by one equation). It can have embedded components at the intersection points of the $\mathbb{P}^1$'s. But $Z(j)^{ord}$ is a divisor on $\hat{\Omega}^{\bullet;ord}_W$. It can have up to two horizontal components. Computing everything is really immediate.

**Example 6.4.** Let $\Lambda = (e_0, e_1)$ be a lattice with basis, let

$$j = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{End}(\Lambda)$$

and define $n$ by $p^n = (a,b,c)$, assume $\alpha > 0$. Then

$$\begin{aligned}\Omega^{ord}_{[\Lambda]} \cap Z(j) =& V(T - \frac{(1+a)T+b}{cT+(1-a)}) \\ =& V(p^n(c'T^2 - 2a'T + b')).\end{aligned}$$

So we see that $\Omega^{ord}_{[\Lambda]}$ occurs with multiplicity $n$ in $Z(j)^{ord}$. If there are horizontal components, these are determined by the second factor of the equation.

Performing a more careful analysis yields the next proposition. Let us write

$$Z(j)^{ord,vert} = \sum_{\Lambda} m_\Lambda(j)\Omega^{ord}_{[\Lambda]}.$$

Let $\mathcal{T}(j)$ denote the fixed point set of $j$ acting on $\mathcal{T}$.

**Proposition 6.5.** *The $m_\Lambda(j)$ can be determined on the Bruhat-Tits tree $\mathcal{T}$ as follows. Write $Q(j) = \varepsilon p^\alpha$ and define $\tilde{j} = j/p^{\lfloor \alpha/2 \rfloor}$. If $\alpha$ is even, then $v_p(Q(\tilde{j})) = 0$ and*

$$\mathcal{T}(\tilde{j}) = \begin{cases} \text{vertex} \\ \text{appartment} \end{cases}.$$

*If $\alpha$ is odd, then $v_p(Q(\tilde{j})) = 1$ and $\mathcal{T}(\tilde{j})$ is an edge. Then $\mathcal{T}(j)$ is a ball of radius $\lfloor \alpha/2 \rfloor$ around $\mathcal{T}(\tilde{j})$. If $\Lambda \in \mathcal{T}(j)$ has distance $i$ of $\mathcal{T}(\tilde{j})$, then $m_{[\Lambda]} = \lfloor \alpha/2 \rfloor - i$.*

**Example 6.6.** If $j = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, then it fixes the chain of (homothety classes) of lattices

$$\ldots \subset (p^2 e_1, e_2) \subset (pe_1, e_2) \subset (e_1, e_2) \subset (p^{-1}e_1, e_2) \subset \ldots.$$

Then $Z(j)$ is the sum of the irreducible components corresponding to these lattices, each with multiplicity 1. There are no vertical components.

There is a picture illustrating the three cases in [11, p. 9].

## 6.3 $p$-adic uniformization of special cycles

Let me first describe $\mathcal{Z}(t)(\mathbb{F})$ in terms of $\hat{\Omega}^\bullet_W(\mathbb{F}) \times I(\mathbb{A}^p_f)/K^p$.

Let $(A,\iota) \in \mathcal{M}(\mathbb{F})$ and choose an isogeny $f : (A_0, \iota_0) \longrightarrow (A,\iota)$. Then $f^{-1}V(A,\iota)_{\mathbb{Q}}f = V(A_0,\iota_0)_{\mathbb{Q}}$. The group $I(A_0)$ acts by conjugation (i.e. pull-back) on $V(A_0,\iota_0)_{\mathbb{Q}}$:

$$s^*x = sxs^{-1}.$$

So set-theoretically,

$$\mathcal{Z}(t)(\mathbb{F})/\cong = \tag{6.1}$$

$$\bigsqcup_{x \in I(A_0)\backslash V(A_0,\iota_0)_{\mathbb{Q}}, Q(x)=t} \{A \mid \exists \text{ quasi-isogeny } f : A_0 \longrightarrow A \text{ s.th. } fxf^{-1} \in \mathrm{End}(A)\}/\cong. \tag{6.2}$$

Now the condition $fxf^{-1} \in \mathrm{End}(A)$ can be checked on Tate modules. Let $f : A_0 \longrightarrow A$ correspond to $(\rho : \mathbb{X} \longrightarrow X, gT^p(A) \subset \mathbb{V}^p(A))$. Then $fxf^{-1} \in \mathrm{End}(A)$ if and only if $\rho \in Z(x)$ and $xgT^p(A_0) \subset gT^p(A_0)$ which is equivalent to $g^{-1}xgT^p(A_0) \subset T^p(A_0)$.

These ideas work in families and yield the $p$-adic uniformization of $\mathcal{Z}(t)$. Let us define

$$\mathcal{C}(t) := \coprod_{x \in V(A_0,\iota_0)_{\mathbb{Q}}, Q(x)=t} Z(x) \times \{gK^p \mid g^{-1}xg \in K^p\} \longrightarrow \mathcal{N} \times I(\mathbb{A}_f^p)/K^p.$$

The group $I(\mathbb{Q})$ acts diagonally on this space. Namely $s \in I(\mathbb{Q})$ defines an isomorphism

$$s : Z(x) \times \{gK^p \mid g^{-1}xg \in K^p\} \to Z(sxs^{-1}) \times \{gK^p \mid g^{-1}sxs^{-1}g \in K^p\}$$
$$(\rho, gK^p) \mapsto (\rho s^{-1}, sgK^p).$$

**Proposition 6.7.** *There is an isomorphism of formal stacks*

$$[I(\mathbb{Q})\backslash\mathcal{C}(t)] \xrightarrow{\cong} \mathcal{Z}(t)_p$$

*which is compatible with the uniformization of $\mathcal{M}_p$.*

# 7 Outlook: Modularity of the Vertical component

The reference is [14][Chap. 4.3].

We now sketch the proof of the modularity of the vertical component of $\hat{\phi}$. Let $Y \subset \mathcal{M}$ be an irreducible component in a fiber of bad reduction. For more details, we refer to [14, 4.3].

**Proposition 7.1.** *The series $\langle \hat{\phi}, (Y,0)\rangle = \langle \hat{\mathcal{Z}}(0,v), (Y,0)\rangle + \log(p) \cdot \sum_{t>0} (\mathcal{Z}(t), Y) q^t \in \mathbb{C}[[q]]$ is the $q$-expansion of a holomorphic modular form of weight $3/2$.*

*Proof.* We will identify $\sum_{t\geq 0}\langle \hat{\mathcal{Z}}(t,v), (Y,0)\rangle$ with a definite theta series of weight 3/2.

Let $V' := V(A_0,\iota_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ with lattice $L := V(A_0,\iota_0)$. Then $V'$ is a 3-dimensional quadratic subspace of $(B^{(p)}, Nm)$, so it is definite.

Let $[\tilde{Y}, g] \subset \hat{\Omega}^{\bullet}_W \times I(\mathbb{A}_f^p)/K^p$ be an irreducible component mapping to $Y \otimes \mathbb{F}$. Then

$$\begin{aligned}(Y, \mathcal{Z}(t))_{\mathcal{M}} &= (Y \otimes \mathbb{F}, \mathcal{Z}(t)_p)_{\mathcal{M}_p} \\ &= ([\tilde{Y}, g], \mathcal{C}(t))_{\hat{\Omega}^{\bullet}_W \times I(\mathbb{A}_f^p)/K^p}\end{aligned}$$

In the last expression, only those components of $\mathcal{C}(t)$ play a role, which lie over $g \in I(\mathbb{A}_f^p)/K^p$. These correspond to $x \in V'$ with $Q(x) = t$ and $g^{-1}xg \in L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$. Define the characteristic function $\varphi^p := 1_{g(L\otimes\hat{\mathbb{Z}}^p)g^{-1}} \in C_c^\infty(V'(\mathbb{A}_f^p))$.

**Lemma 7.2** (see [14, Proposition 4.3.2])**.** *There exists a function $\mu_{\tilde{Y}} \in C_c^\infty(V'(\mathbb{Q}_p))$ such that for all $x \in V'(\mathbb{Q}_p)$,*

$$\mu_{\tilde{Y}}(x) = (\tilde{Y}, Z(x))_{\hat{\Omega}^{\bullet}_W}.$$

Performing an extra anlysis of $\langle \hat{\mathcal{Z}}(0,v), (Y,0)\rangle$ and $\mu(0)$ yields that

$$\sum_{t\geq 0}\langle \hat{\mathcal{Z}}(t,v), (Y,0)\rangle = \sum_{x\in V'(\mathbb{Q})} (\varphi^p \otimes \mu_{\tilde{Y}})(x) q^{Q(x)}.$$

The result follows. $\square$

# References


[1] Deligne, P., Mumford, D.: *On the irreducibility of the moduli space of curves of genus* $\geq 2$.

[2] ARGOS Seminar in Bonn, *Intersections of Modular Correspondences*, Astérisque, **312**, Soc. Math. France, Paris, 2007.

[3] Bosch, S., Lütkebohmert, W., Raynaud, M.: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1990.

[4] R. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J., **97** (1999), 219-233.

[5] J.-B. Bost, *Potential theory and Lefschetz theorems for arithmetic surfaces*, Ann. Sci. École Norm. Sup. **32** (1999), 241-312.

[6] J.-F. Boutot, H. Carayol, *Uniformisation p-adique des courbes de Shimura*, Courbes Modulaires et Courbes de Shimura, Astérisque, **196-197**, 45-158, Soc. Math. France, Paris, 1991.

[7] Demazure, M.: *Lectures on p-divisible groups*, Lecture Notes in Mathematics **302**, Springer, 1972.

[8] B. Gross, *On canonical and quasi-canonical liftings*, Invent. Math. **84** (1986), 321-326.

[9] B. Howard, *Moduli spaces of CM elliptic curves and derivatives of Eisenstein series*, Lecture notes for lectures in 2009 at Morningside Center for Mathematics.

[10] S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Annals of Math., **146** (1997), 545-646.

[11] S. Kudla, M. Rapoport, *Height pairings on Shimura curves and p-adic uniformization*, Invent. Math. **142** (2000), no. 1, 153-223.

[12] S. Kudla, M. Rapoport, T. Yang, *On the derivative of an Eisenstein series of weight one*, Internat. Math. Res. Notices 1999, no. 7, 347-385.

[13] S. Kudla, M. Rapoport, T. Yang, *Derivatives of Eisenstein series and Faltings heights*, Compos. Math. **140** (2004), no. 4, 887-951.

[14] S. Kudla, M. Rapoport, T. Yang, *Modular forms and special cycles on Shimura curves*, Annals of Mathematics Studies Vol. 161.

[15] G. Laumon, L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **39**, Springer-Verlag, Berlin, 2000.

[16] M. Rapoport, T. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, **141**, Princeton University Press, Princeton, 1996.

[17] M. Raynaud, *Hauteurs et isogénies*, Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell, Astérisque, **127**, 199-234, Soc. Math. France, Paris, 1985.

[18] C. Soulé, *Lectures on Arakelov theory*, Cambridge Studies in Advanced Math. **33**, Cambridge Univ. Press, Cambridge, 1992.

[19] E. Viehmann, K. Ziegler, *Formal moduli of formal* $\mathcal{O}_K$*-modules*, in [2], p. 57-66.

[20] S. Wewers, *Canonical and quasi-canonical liftings*, in [2], p. 67-86.