Solutions for exercises, Algebra I (Commutative Algebra) – Week 9

Exercise 43. (Noether normalization over rings, 3 points)
Notice that A, being a subring of an integral domain, is a integral domain.
By assumption there is a surjective homomorphism of A-algebras: $f : A[x_1, \dots, x_n] \twoheadrightarrow B$. We can localize f with respect to the multiplicative set $S = A\backslash\{0\}$ (i.e. tensor with $Q(A)$) to get a surjective homomorphism of $Q(A)$ algebras: $S^{-1}(f) : Q(A)[x_1, \dots, x_n] \twoheadrightarrow S^{-1}B$. In particular, $S^{-1}B$ is a $Q(A)$-algebra of finite type. Thus by Noether normalization theorem there are $b_1, \dots, b_k \in S^{-1}B$ such that the homomorphism of $Q(A)$-algebras $g : Q(A)[X_1, \dots, X_k] \to S^{-1}B$, $X_i \mapsto \frac{b_i}{a_i}$ gives an isomorphism $Q(A)[X_1, \dots, X_k] \simeq Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}]$ and $S^{-1}B$ is a finite $Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}]$-module. In particular $S^{-1}B$ is integral over $Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}]$.
Set $c_i = f(x_i)$ for $i = 1, \dots, n$. As $S^{-1}B$ is integral over $Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}]$, for any i, $\frac{c_i}{1} \in S^{-1}B$ is annihilated by a (monic) polynomial $P_{c_i} \in Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}][x]$. If $0 \neq a \in A$ is the product of $(a_1 \cdots a_k)^d$ (where $d = \max_i(\deg(P_{c_i}))$) by the product of all the denominators of the coefficients of the P_i's, we have that $0 \neq aP_{c_i} \in A[b_1, \dots, b_k][x]$ and $aP_{c_i}(c_i) = 0$. Then $P_{c_i} \in A_a[b_1, \dots, b_k][x]$ for any i i.e. c_i is integral over $A_a[b_1, \dots, b_k]$ for any i i.e. $A_a[b_1, \dots, b_k][c_1, \dots, c_n]$ is a finite $A_a[b_1, \dots, b_k]$-module. Tensoring f with A_a, we see that $A_a[c_1, \dots, c_n] = B \otimes_A A_a \simeq B_a$; a fortiori $A_a[b_1, \dots, b_k][c_1, \dots, c_n] \simeq B_a$. Thus B_a is integral over $A_a[b_1, \dots, b_k]$ and since $\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}$ were algebraically independent over $Q(A)$, $b_1, \dots, b_k$ are algebraically independent over A (indeed, because A is an integral domain, $\ker(A[X_1, \dots, X_k] \to A[b_1, \dots, b_k]) \hookrightarrow \ker(Q(A)[X_1, \dots, X_k] \to Q(A)[\frac{b_1}{a_1}, \dots, \frac{b_k}{a_k}]) = \{0\}$).

Exercise 44. (Finite type $\mathbb{Z}$-algebras are Jacobson, 3 points)
Notice first that the quotient of a Jacobson ring is Jacobson: indeed the ideals of $A/\mathfrak{a}$ correspond exactly to the ideals of A containing $\mathfrak{a}$. So if $\mathfrak{q} \in \operatorname{Spec}(A/\mathfrak{a})$ then $\mathfrak{p} = \mathfrak{q}^c \in V(\mathfrak{a})$ can be written $\mathfrak{p} = \cap_{\mathfrak{p} \subsetneq \mathfrak{m} \in \operatorname{MaxSpec}(A)} \mathfrak{m}$; thus passing to the quotient we get $\mathfrak{q} = \cap_{\mathfrak{q} \subsetneq \mathfrak{m} \in \operatorname{MaxSpec}(A)} \mathfrak{m}$ (since $A/\mathfrak{a}/\mathfrak{m}/\mathfrak{a} \simeq A/\mathfrak{m}$ a field).

Assume first that B is integral over A and (A Jacobson). By the above observation, we can assume that $A \subset B$ with A Jacobson and B integral over A. Let $\mathfrak{q} \in \operatorname{Spec}(B)$ (not maximal) and $\operatorname{Spec}(A) \ni \mathfrak{p} = \mathfrak{q}^c = A \cap \mathfrak{q}$ (not maximal neither by the 4^{th} step of the proof of the Going-up theorem). By hypothesis $\mathfrak{p} = \cap_{\mathfrak{p} \subsetneq \mathfrak{m} \in \operatorname{MaxSpec}(A)} \mathfrak{m}$. Since B is integral over A, by the Going-up theorem, for any $\mathfrak{p} \subset \mathfrak{m}$ there is a $\mathfrak{q} \subset \mathfrak{n} \in \operatorname{Spec}(B)$ such that $\mathfrak{n} \cap A = \mathfrak{m}$. By the first step of the proof the Going-up theorem, $B/\mathfrak{n}$ is integral over $A/\mathfrak{m}$; and by the third step of the same proof, since $A/\mathfrak{m}$ is a field, $B/\mathfrak{n}$ is also a field i.e. such a $\mathfrak{n}$ is maximal. Set $\mathfrak{b} = \cap_{\mathfrak{n} \in \operatorname{MaxSpec}(B),\ \mathfrak{q} \subsetneq \mathfrak{n} \text{ and } \mathfrak{p} \subset \mathfrak{n} \cap A \in \operatorname{MaxSpec}(A)} \mathfrak{n} = \cap_{\mathfrak{n} \in \operatorname{MaxSpec}(B),\ \mathfrak{q} \subsetneq \mathfrak{n}} \mathfrak{n}$ (by the 4^{th}-step of the Going-up theorem $\mathfrak{n} \cap A$ is maximal). We have $\mathfrak{q} \subset \mathfrak{b}$ and $\mathfrak{b} \cap A = \cap_{\mathfrak{n} \in \operatorname{MaxSpec}(B),\ \mathfrak{q} \subset \mathfrak{n}} \mathfrak{n} \cap A = \cap_{\mathfrak{p} \subsetneq \mathfrak{m} \in \operatorname{MaxSpec}(A)} \mathfrak{m} = \mathfrak{p} = \mathfrak{q} \cap A$. We adapt the proof of the 5^{th} step of the proof of the Going-up to conclude that $\mathfrak{q} = \mathfrak{b} = \cap_{\mathfrak{n} \in \operatorname{MaxSpec}(B),\ \mathfrak{q} \subsetneq \mathfrak{n}} \mathfrak{n}$. Thus B is Jacobson.

Let us prove this characterization of Jacobson ring: A is Jacobson if and only if for any prime $\mathfrak{p} \subset A$ for which there is a $0 \neq a \in A/\mathfrak{p}$ such that $(A/\mathfrak{p})_a$ is a field, then $A/\mathfrak{p}$ is a field: assume A is Jacobson. Then $A/\mathfrak{p}$ is an integral domain which is Jacobson (first remark). If

Solutions to be handed in before Tuesday June 15, 4pm.

$(A/\mathfrak{p})_a$ is a field we have $(0) = \mathrm{Spec}((A/\mathfrak{p})_a) = \{\mathfrak{q} \in \mathrm{Spec}(A/\mathfrak{p}),\ a \notin \mathfrak{q}\}$ so if $A/\mathfrak{p}$ contains a non-zero prime ideal we have $a \in \cap_{(0)\neq\mathfrak{q}}\mathfrak{q}$ but since $A/\mathfrak{p}$ is Jacobson (and an integral domain) $\cap_{(0)\neq\mathfrak{q}}\mathfrak{q} = \mathfrak{N}_{A/\mathfrak{p}} = (0)$ i.e. $a = 0$. Contradiction. So $\mathrm{Spec}(A/\mathfrak{p}) = (0)$ i.e. $A/\mathfrak{p}$ is a field.
Conversely if $\mathfrak{p} \in \mathrm{Spec}(A)$, denote $\mathfrak{a} = \cap_{\mathfrak{p}\subsetneq\mathfrak{m}\in\mathrm{MaxSpec}(A)}\mathfrak{m}$. If $\mathfrak{p} \subsetneq \mathfrak{a}$, pick a $a \in \mathfrak{a}\backslash\mathfrak{p}$; let us consider a prime ideal $\mathfrak{q}$ which is maximal among those containing $\mathfrak{p}$ and not containing a. By definition of $\mathfrak{a}$, $\mathfrak{q}$ is not a maximal ideal of A but $\{a^n,\ n \geq 0\}^{-1}\mathfrak{q}$ is a maximal ideal of A_a. So $A_a/\{a^n,\ n \geq 0\}^{-1}\mathfrak{q} \simeq (A/\mathfrak{q})_a$ is a field. Thus $A/\mathfrak{q}$ is a field i.e. $\mathfrak{q}$ is maximal. Contradiction. So $\mathfrak{p} = \mathfrak{a}$.

Let us prove that if A is Jacobson then any ring which is generated by one element as a A-algebra (i.e. a quotient of $A[x]$) is also Jacobson: let $C = A[x]/\mathfrak{a}$ be such a ring and let $\mathfrak{p} \in V(\mathfrak{a}) \subset \mathrm{Spec}(A[x])$, and consider the quotient homomorphism $f : C \to C/\mathfrak{p} \simeq A[x]/\mathfrak{p}$. We must show that if $0 \neq a \in A[x]/\mathfrak{p}$ is such that $(A/\mathfrak{p})_a$ is a field then $(A/\mathfrak{p})$ is also a field. Let us denote $B = f(A) \subset A[x]/\mathfrak{p}$. By the first remark B is Jacobson and an integral domain (as subring of an integral domain) so $\cap_{\mathfrak{m}\in\mathrm{MaxSpec}(B)}\mathfrak{m} = (0)$. Look at $B[x] \twoheadrightarrow A[x]/\mathfrak{p}$ $(x \mapsto \overline{x})$. If it is an isomorphism, and if $0 \neq a \in A[x]/\mathfrak{p}$ is such that $(A/\mathfrak{p})_a$ is a field, then $B[x]_{\overline{a}}$ is a field. But then $Q(B)[x]_{\overline{a}}$ is also a field. But looking at the description of the prime ideals of the principal ideal domain $Q(B)[x]$ we see that it is Jacobson; thus the fact that $Q(B)[x]_{\overline{a}}$ is a field implies that $Q(B)[x]$ is a field. Contradiction. So $B[x] \twoheadrightarrow A[x]/\mathfrak{p}$ is not an isomorphism and $A[x]/\mathfrak{p} \simeq B[x]/\mathfrak{q}$ for a non-zero prime ideal ($\mathfrak{q} = \ker(B[x] \twoheadrightarrow A[x]/\mathfrak{p})$ and $A[x]/\mathfrak{p}$ is an integral domain). If $0 \neq a \in B[x]/\mathfrak{q}$ is such that $(B[x]/\mathfrak{q})_a$ is a field.
If $g \in \mathfrak{q}$ is a non-zero polynomial with leading coefficient $d \in B$, then $\overline{x}$ is integral over B_d. So $B[x]/\mathfrak{q}$ is integral over B_d. In particular as $a \in B[x]/\mathfrak{q}$, there is a monic polynomial $h = y^n + h_1 y^{n-1} + \cdots + h_{n-1} \in B_d[y]$ (with $h(0) \neq 0$ because B is an integral domain) such that $h(a) = 0$. So dividing by $h_{n-1}a^n$ we find $a^{-n} + \frac{h_{n-2}}{h_{n-1}}a^{-(n-1)} + \cdots + \frac{1}{h_{n-1}} = 0$ i.e. a^{-1} is integral over $B_{h_{n-1}d}$. So $(B[x]/\mathfrak{q})_a$ is integral over $B_{h_{n-1}d}$. By the 3^{rd} step of the proof of the Going-up theorem, $B_{h_{n-1}d}$ is also a field. But since B is Jacobson, (and (0) is prime) B is a field. In particular $B = B_{hn-1d}$. So $B[x]/\mathfrak{q} \subset (B[x]/\mathfrak{q})_a$ (since $B[x]/\mathfrak{q}$ is an integral domain) is integral over the field B. Again by the 3^{rd} step of the proof of the Going-up theorem, $B[x]/\mathfrak{q}$ is a field. So $B[x]/\mathfrak{q} \simeq A[x]/\mathfrak{p}$ is Jacobson. In particular $(0) = \cap_{\mathfrak{m}\in\mathrm{MaxSpec}(A[x]/\mathfrak{p})}\mathfrak{m}$ i.e. $\mathfrak{p}/\mathfrak{a} = \cap_{\mathfrak{p}/\mathfrak{a}\subset\mathfrak{m}\in\mathrm{MaxSpec}(C)}\mathfrak{m}$.

For an A-algebra generated by finitely many elements, we proceed by induction.

Exercise 45. (Finite fields, 3 points)
Assume k is a field which is a finitely generated $\mathbb{Z}$-algebra. If the natural homomorphism is injective $\mathbb{Z} \hookrightarrow k$ then by the universal property of localization with have a field extension $\mathbb{Q} \hookrightarrow k$ and k is a fortiori a $\mathbb{Q}$-algebra of finite type. By Noether normalization, there are a $\ell \geq 0$ and an injective homomorphism $\mathbb{Q}[x_1, \ldots x_\ell] \hookrightarrow k$ such that k is a finite $\mathbb{Q}[x_1, \ldots, x_\ell]$. By Corollary 11.11 k is integral over $\mathbb{Q}[x_1, \ldots, x_\ell]$. By the 3^{rd} step of the proof of the Going-up theorem $\mathbb{Q}[x_1, \ldots, x_\ell]$ is a field i.e. $\ell = 0$. Thus k is a finite field extension of $\mathbb{Q}$ (i.e. a number field).
Let us prove that a number field cannot be a finitely generated $\mathbb{Z}$-algebra: let $f : \mathbb{Z}[x_1, \ldots, x_n] \to k$ be a ring homomorphism and let us denote $\alpha_i = f(x_i) \in k$. Let $\ell \in \mathbb{Z}_{>0}$ be the product of all the denominators of the minimal polynomials of α_i over $\mathbb{Q}$. Then the minimal polynomials of the α_i's are in $\mathbb{Z}_\ell[x]$ i.e. k is integral over $\mathbb{Z}_\ell$. So by the 3^{rd} step of the proof of the Going-up theorem $\mathbb{Z}_\ell$ is a field; which is impossible (any prime not dividing ℓ is not invertible in $\mathbb{Z}_\ell$).

So the homomorphism $\mathbb{Z} \to k$ is not injective; thus there is a prime number $p > 0$, such that the homomorphism factors through $\mathbb{F}_p$. So k is in particular a $\mathbb{F}_p$-algebra of finite type. By Noether normalization k is a finite module over a polynomial ring over $\mathbb{F}_p$, in particular it is integral over a polynomial ring. Again by the 3^{rd} step of the proof of the Going-up theorem, k is a finite field extension of $\mathbb{F}_p$ i.e. a finite field.

Exercise 46. (Family of polynomials without common zeros, 3 points)
Using Remark 12.11: since $Z((f_1,\dots,f_k)) = \emptyset$ we have $\sqrt{(f_1,\dots,f_k)} = I(Z((f_1,\dots,f_k))) = \mathbb{C}[x_1,\dots,x_n]$. So $1 \in \sqrt{(f_1,\dots,f_k)}$ i.e. $1^n = 1 \in (f_1,\dots,f_k)\otimes\mathbb{C}$.
If $(f_1,\dots,f_k) = \mathbb{Z}[x_1,\dots,x_n]$ we are done. So we can assume that $(f_1,\dots,f_k) \subsetneq \mathbb{Z}[x_1,\dots,x_n]$ there is a maximal ideal $(f_1,\dots,f_k) \subset \mathfrak{m}$ containing it. We have an exact sequence:

$$0 \to \mathfrak{m} \to \mathbb{Z}[x_1,\dots,x_n] \to k \to 0$$

where k is the quotient field. The sequence also shows that k is finitely generated $\mathbb{Z}$-algebra hence, by the previous exercise, k is a finite field, of characteristic, say $p > 0$.
Since $\mathbb{C}$ is a flat $\mathbb{Z}$-algebra (we have seen that $\mathbb{Q}$ is a $\mathbb{Z}$-algebra and $\mathbb{C}$ is a $\mathbb{Q}$-vector space (i.e. a free $\mathbb{Q}$-module)), we have $\mathbb{C}[x_1,\dots,x_n] = (f_1,\dots,f_k)\otimes\mathbb{C} \subset \mathfrak{m}\otimes\mathbb{C}$. So we get $(f_1,\dots,f_k)\otimes\mathbb{Q} = \mathfrak{m}\otimes\mathbb{Q}$ thus any element of $\mathfrak{m}/(f_1,\dots,f_k)$ is annihilated by an integers.

Now, $p \in \mathbb{Z}[x_1,\dots,x_n]$ is sent to 0 in k i.e. $p \in \mathfrak{m}$. As $\mathfrak{m}/(f_1,\dots,f_k)$ is torsion, there is a $d \in \mathbb{Z}\backslash\{0\}$, such that $0 \neq dp \in (f_1,\dots,f_k)$; which proves the result.

The result does not hold if $\mathbb{C}$ is replaced by $\mathbb{R}$: for example $x^2+1 \in \mathbb{Z}[x]$ has no real zero but the principal ideal (x^2+1) does not contain a non-zero integer (for degree reason).

Exercise 47. (Noether normalization via linear projections, 4 points)
We notice that when x is fixed $x = a$, the system of equations $y - z^2 = 0$; $az - y^2 = 0$ transforms into $y - z^2 = 0$; $(a - z^3)z = 0$ which admits finitely many solutions. So let us consider the projection on the x-axis.
Let us denote $A = k[x,y,z]/\mathfrak{a}$ and consider the composition $f : k[x] \to A$ of the inclusion $k[x] \hookrightarrow k[x,y,z]$ and the canonical projection $k[x,y,z] \twoheadrightarrow k[x,y,z]/\mathfrak{a}$.
If $P \in \ker(f)$ then $P \in (y - z^2, xz - y^2)$ i.e. $P = (y - z^2)p(x,y,z) + (xz - y^2)q(x,y,z)$ for some $p, q \in k[x,y,z]$. But looking at $y = 0 = z$ we get $P = 0$ i.e. f is injective.
We claim that $1, z, z^2, z^3$ generate A as a $k[x]$-module: because of the surjection $k[x][y,z] \twoheadrightarrow A$, y, z generate A as a $k[x]$-algebra. In A, $\overline{y} = \overline{z}^2$ thus $\overline{z}$ generates A as a $k[x]$-algebra. Moreover $\overline{z}^4 = \overline{xz}$ in A; thus any polynomial $p \in k[x,y,z]$ is in the class modulo $\mathfrak{a}$ of a polynomial whose monomials are of the form $x^k z^i$, $k \in \mathbb{N}$, $i \in \{0,1,2,3\}$; which proves the claim.
So A is a finite $k[x]$-algebra and as such it is integral over $k[x]$ (Corollary 11.11). So by the Going-up theorem (Theorem 11.33), $\varphi : \operatorname{Spec}(A) = V(\mathfrak{a}) \to \operatorname{Spec}(k[x]) = \mathbb{A}^1_k$ is surjective and by Remark 11.35 (i) it is closed (alternatively remark that φ has the going-up property by going-up theorem and since A is Noetherian (as quotient of the Noetherian ring $k[x,y,z]$), Exercise 38 yields that φ is closed).
For $\mathfrak{p} \in \operatorname{Spec}(k[x])$, we have seen in (the solution of) Exercise 37 (ii) that the fiber $\varphi^{-1}(\mathfrak{p})$ of φ over $\mathfrak{p}$ is isomorphic to $\operatorname{Spec}(A \otimes_{k[x]} Q(k[x]/\mathfrak{p}))$. Since A is a finite $k[x]$-module (i.e. there is a surjective homomorphism of $k[x]$-modules $k[x]^4 \twoheadrightarrow A$), $A \otimes Q(k[x]/\mathfrak{p})$ is a finite $Q(k[x]/\mathfrak{p})$-algebra in particular $A \otimes Q(k[x]/\mathfrak{p})$ is a finite-dimensional $Q(k[x]/\mathfrak{p})$-vector space.

Any prime ideal of $A \otimes Q(k[x]/\mathfrak{p})$ is maximal: a prime ideal $\mathfrak{q} \in \operatorname{Spec}(A \otimes Q(k[x]/\mathfrak{p}))$ is in particular a $Q(k[x]/\mathfrak{p})$-vector subspace of $A \otimes Q(k[x]/\mathfrak{p})$ so the integral domain $B = A \otimes Q(k[x]/\mathfrak{p})/\mathfrak{q}$ is also a finite-dimensional $Q(k[x]/\mathfrak{p})$-vector space (as quotient of finite-dimensional vector space). Now take $x \in B\backslash\{0\}$ and consider the $Q(k[x]/\mathfrak{p})$-linear map $m_x : B \to B$, $b \mapsto bx$. Since B is an integral domain, m_x is injective and since B is finite-dimensional, the linear map m_x is also surjective. In particular $1 \in \operatorname{im}(m_x)$ i.e. there is a $y \in B$ such that $yx = 1$ i.e. x is a unit. So B is a field i.e. $\mathfrak{q}$ is maximal.

As $A \otimes Q(k[x]/\mathfrak{p})$ is a finite-dimensional $Q(k[x]/\mathfrak{p})$-vector space (and ideals of $A \otimes Q(k[x]/\mathfrak{p})$ are in particular $Q(k[x]/\mathfrak{p})$-vector subspaces), $A \otimes Q(k[x]/\mathfrak{p})$ is Noetherian. So as seen in (solution for Exercise 38) $\mathrm{Spec}(A \otimes Q(k[x]/\mathfrak{p}))$ can be written as a finite union $\mathrm{Spec}(A \otimes Q(k[x]/\mathfrak{p})) = \cup_{i=1}^{n} V(\mathfrak{q}_i)$ where $\mathfrak{q}_i \in \mathrm{Spec}(A \otimes Q(k[x]/\mathfrak{p}))$. Since any prime ideal in $A \otimes Q(k[x]/\mathfrak{p})$ is maximal we get $\mathrm{Spec}(A \otimes Q(k[x]/\mathfrak{p})) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ i.e. any fiber of φ is finite.

Exercise 48. (Valuation rings, 3 points)

1. Since A is a subring of a field, it is an integral domain and since $A \subset K$ the universal property of localization gives the inclusions $A \subset Q(A) \subset K$. Now let $a \in K \subset L$; then either $a \in B$, in which case $a \in B \cap K = A$, or $a^{-1} \in B$, in which case $a^{-1} \in B \cap K = A$. Since $Q(A) \subset K$, this proves that the same property holds for $Q(A)$ i.e. that A is a valuation ring. It also proves that $Q(A) \subset K$ is surjective (hence an isomorphism) since if $a \in K \backslash A$ then $a^{-1} \in A$; so $a = (a^{-1})^{-1} \in Q(A)$.

2. Assume A is a field and L/K is algebraic. By the first question we get $A = Q(A) = K$. In particular $K \subset B$. Let $b \in B$; as $b^{-1} \in L$ is algebraic over K, take $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} \in K[x] \backslash \{0\}$ such that $f(b^{-1}) = 0$. Taking the product of the equality $b^{-n} = -(a_1 b^{-(n-1)} + \cdots + a_{n-1}) \in L$ by b^{n-1}, we get $b^{-1} = -(a_1 + a_2 b + \cdots + a_{n-1} b^{n-1})$ i.e. $(K \subset B)$ $b^{-1} \in B$. Therefore B is a field.