Exercise 30. (i) Let \( \varphi : (0, \infty) \to \mathbb{C} \) be a function such that there exists \( l > 1 \) with \( |\varphi(y)| \leq y^l \) for all \( y > 0 \). Show that then for \( z \in \mathbb{H} \) the series

\[
\Theta(\varphi)(z) := \sum_{\gamma \in \Gamma \setminus \Gamma_0} \varphi(\Im(\gamma z))
\]

converges absolutely. (In this exercise \( \Gamma = \text{SL}_2(\mathbb{Z}) \).) Moreover, show that if \( \varphi \) is compactly supported, all but finitely many terms vanish.

(ii) Let \( f : \Gamma \setminus \mathbb{H} \to \mathbb{C} \) be a square-integrable function with Fourier expansion \( f(x + iy) = \sum_{n \in \mathbb{Z}} c_n(y) e^{2\pi i nx} \). Suppose that \( \varphi \) is such that \( \Theta(\varphi) \in L^2(\Gamma \setminus \mathbb{H}) \). Show that

\[
\langle \Theta(\varphi), f \rangle = \int_0^\infty \varphi(y) c_0(y) y^{-2} \, dy,
\]

where \( \langle f, g \rangle = \int_{\Gamma \setminus \mathbb{H}} f(z) \overline{g(z)} y^{-2} \, dx \, dy \) denotes the usual inner product on \( L^2(\Gamma \setminus \mathbb{H}) \).

Exercise 31. Let \( s \in \mathbb{C} \). Show that there exists \( N > 0 \) such that the following holds: If \( f \) is a Maass form with eigenvalue \( s(1 - s) \) and Fourier expansion

\[
f(z) = \sum_{n : |n| \geq N} a_n y^{1/2} K_{s-1/2}(2\pi |n| y) e^{2\pi i nx},
\]

then \( f(z) = 0 \) for all \( z \in \mathbb{H} \).

Exercise 32. (i) Show that for \( \Re z > 0 \) and \( \Re \nu > 0 \) the \( K \)-Bessel function \( K_\nu(z) \) can be written as

\[
K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-\frac{z}{2}(u+n^{-1})} u^{\nu-1} \, du.
\]

(ii) Prove that if \( \nu \) with \( \Re \nu > 0 \) is fixed and \( \Re z > 0 \), then

\[
K_\nu(z) \sim 2^{\nu-1} \Gamma(\nu) z^{-\nu}
\]

as \( z \to 0 \).

(iii) Let \( r, s \) be real numbers. Show that for \( \Re \nu > 0 \) and \( s > 1/2 \) we have

\[
\int_{-\infty}^\infty (s^2 + u^2)^{-\nu} e^{2iru} \, du = \begin{cases} 
\frac{2 \Gamma(1/2) \Gamma(1/2) |s|^{\nu-\frac{1}{2}}}{\Gamma(\nu)} K_{\nu-\frac{1}{2}}(2|rs|) & \text{if } r \neq 0, \\
|s|^{-2\nu} \frac{\Gamma(1/2) \Gamma(\nu-1/2)}{\Gamma(\nu)} & \text{if } r = 0.
\end{cases}
\]

(Hint: Multiply by \( \Gamma(\nu) \) and use \( \Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} \, dt \).)
Exercise 33. Let $G = \text{Sl}_2(\mathbb{R})$ and $K = \text{SO}(2)$. Let $\mathcal{S}$ denote the set of positive definite symmetric matrices in $G$.

(i) Show that the maps

$$G/K \rightarrow \mathcal{S}, \ gK \mapsto gg^t,$$

and

$$\mathbb{H} \rightarrow \mathcal{S}, \ z = x + iy \mapsto y^{-1} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} =: A_z,$$

are bijections and preserve the action of $G$ on the respective spaces.

(ii) For $s \in \mathbb{C}$ with $\Re s > 1$ and $Y \in \mathcal{S}$ define $Z(Y, s) = \frac{1}{2} \sum_{v \in \mathbb{Z}^2 \setminus \{0\}} (v^t Y v)^{-s}$ with $v^t$ denoting the transpose of $v$. Prove that

$$\zeta(2s) E_\infty(z, s) = Z(A_z, s),$$

where $E_\infty(z, s)$ is the (non-analytic) Eisenstein series introduced in the lectures.