Exercise 27. For \( z \in \mathbb{H} \) let \( q = e^{2\pi iz} \), and define
\[
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]

(i) Show that the infinite product defining \( \eta \) converges absolutely for every \( z \in \mathbb{H} \).

(ii) Put \( f(z) := \sum_{m,n=1}^{\infty} m^{-1} q^{mn} \), and let \( L_f(s) \) be the \( L \)-function attached to its Fourier expansion for sufficiently large \( \Re s \). Show that \( \Lambda_f(s) := (2\pi)^{-s} \Gamma(s) L_f(s) \) can be meromorphically continued to all \( s \in \mathbb{C} \), satisfies \( \Lambda_f(s) = \Lambda_f(-s) \), and that
\[
\Lambda_f(s) + \frac{1}{2s^2} - \frac{\pi}{12(s-1)} + \frac{\pi}{12(s+1)}
\]
is everywhere holomorphic and bounded on the vertical strips \( \alpha \leq \Re s \leq \beta \) for every \(-\infty < \alpha < \beta < \infty \).

(\textit{Hint: Express} \( L_f(s) \) \textit{in terms of the Riemann zeta function}.)

(iii) Prove that for every \( y \in \mathbb{R}_{>0} \),
\[
f(iy) = f(-\frac{1}{iy}) + \frac{\pi}{12y} + \frac{1}{2} \log y - \frac{\pi y}{12}.
\]

(\textit{Hint: Use Mellin inversion}.)

(iv) Conclude that for all \( z \in \mathbb{H} \),
\[
\eta(-\frac{1}{z}) = \left( \frac{z}{i} \right)^{1/2} \eta(z),
\]
i.e., \( \eta \) is a modular form in \( M(1, \frac{1}{2}, 1) \). Further show that \( \Delta = (2\pi)^{12} \eta^{24} \).

Exercise 28. Prove that
\[
q^{1/24} \eta(z)^{-1} = \sum_{n=0}^{\infty} p(n) q^n,
\]
where \( p(n) \in \mathbb{N} \) is the number of partitions of \( n \), i.e. the number of unordered tuples \( (k_1, \ldots, k_n) \), with \( k_1, \ldots, k_n \in \mathbb{N}_0 \) and \( k_1 + \ldots + k_n = n \). (We set \( p(0) := 1 \).)

Exercise 29. For \( z \in \mathbb{H}^* \) let \( j(z) := \frac{(2\pi)^{12} G_4(z)^3}{\Delta(z)} \). Prove the following properties of \( j \):
(i) \( j \) is a meromorphic function on \( \mathbb{H}^* \) and holomorphic on \( \mathbb{H} \).

(ii) \( j \) has a Fourier expansion \( j(z) = q^{-1} + \sum_{n \geq 0} a_n q^n \) at \( \infty \) with \( a_n \in \mathbb{Z} \).

(iii) \( j(i) = 1728 \) and \( j(\rho) = 0 \) for \( \rho \) a third root of unity. (Hint: First show that \( G_4(\rho) = G_6(i) = 0 \).)

(iv) \( j \) defines a bijection \( \Gamma(1) \backslash \mathbb{H}^* \to \mathbb{P}^1_{\mathbb{C}} \) for \( \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) the complex projective line. Explain how this gives rise to a bijection from the isomorphism classes of complex elliptic curves to the complex numbers in \( \mathbb{C} \).