Advanced Global Analysis I

Exercises 5

November 20, 2013

Due: November 28, 2013

Exercise 19. For \( k \in \mathbb{Z}, \ k \geq 2 \), let
\[
G_{2k}(z) = \sum_{(m,n) \neq (0,0)} (mz + n)^{-2k}.
\]
Define \( \Delta(z) = (60G_4(z))^3 - 27(140G_6(z))^2 \).

(i) Show that \( \Delta \) is a cusp form of weight 12 with respect to \( \text{SL}_2(\mathbb{Z}) \).

(ii) Write \( \Delta(z) = (2\pi)^{12} \sum_{n \geq 1} \tau(n) e^{2\pi i nz} \) for the Fourier expansion of \( \Delta \). Prove that the coefficients \( \tau(n) \) are integers for all \( n \), and that they satisfy the relation \( \tau(mn) = \tau(m)\tau(n) \) if \( n, m \) are coprime.

(iii) Compute the dimension of the space of modular forms of weight \( k \) for every \( k \geq 0 \).

Exercise 20. Let \( n, m > 0 \) be integers such that \( m \) is a divisor of \( n \).

(i) Show that the set \( \Gamma(1) \langle \binom{a}{c} \mid \text{det}(\binom{a}{c}) = nm, \gcd(a,b,c,d) = m \rangle \Gamma(1) \) consists of the matrices \( \binom{a}{c} \in \text{Mat}_{2 \times 2}(\mathbb{Z}) \) of determinant \( nm \) for which the greatest common divisor of \( a, b, c, d \) is equal to \( m \). (Recall that \( \Gamma(1) = \text{SL}_2(\mathbb{Z}) \).)

(ii) Show that \( \Gamma(1) \langle \binom{n}{m} \rangle \Gamma(1) = \bigcup_{a,d > 0, \ ad = nm, \ 0 \leq b < d, \ \gcd(a,b,d) = m} \Gamma(1) \langle \binom{a}{0} \binom{b}{d} \rangle \) and that this union is disjoint.

Exercise 21. Suppose \( n > 0 \) is an integer which is divisible by 8 and let \( A \in \text{SL}_n(\mathbb{Z}) \) be a positive definite symmetric matrix satisfying \( \langle v^t Av \rangle \equiv 0 \mod 2 \) for all \( v \in \mathbb{Z}^n \).

(i) Use the Poisson summation formula to show that \( \theta(A,z) \) is a holomorphic automorphic form of weight \( n/2 \) with respect to \( \text{SL}_2(\mathbb{Z}) \), where
\[
\theta(A,z) := \sum_{v \in \mathbb{Z}^n} e^{\pi i (v^t Av)z}.
\]

(ii) Compute \( c_0 \) such that \( \theta(A,z) - c_0 G_{n/2}(z) \) is a cusp form.

(iii) For a non-negative integer \( m \) let
\[
r_A(m) = \left| \{ v \in \mathbb{Z}^n \mid v^t Av = 2m \} \right|.
\]
Show that there is a constant \( C > 0 \) such that
\[
\left| r_A(m) - \frac{(2\pi)^{n/2}}{\zeta(n/2)(n/2 - 1)!} \sigma_{n/2 - 1}(m) \right| \leq Cm^{n/4}
\]
for all \( m \geq 1 \).