Exercise 16. For \( k \in \mathbb{Z}, k \geq 2 \), let \( G_{2k}(z) = \sum_{(m,n) \neq (0,0)} (mz + n)^{-2k} \). Show that:

(i) \( G_{2k}(z) = 2\zeta(2k) \sum_{\gamma \in B \setminus \text{Sl}_2(\mathbb{Z})} \left( \frac{d(\gamma z)}{dz} \right)^k \)

where \( \zeta \) is the Riemann zeta function and \( B = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}_2(\mathbb{Z}) \mid c = 0 \} \).

(ii) The Fourier expansion of \( G_{2k} \) is

\[ G_{2k}(z) = 2\zeta(2k) + 2 \left( \frac{2\pi i}{2k - 1} \right)^{2k} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi inz}, \]

where \( \sigma_k(n) = \sum_{m \in \mathbb{N}, m | n} m^k \).

(Hint: Use the cotangent identity \( \pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} + \frac{1}{z-n} \right) \).)

(iii) \( G_{2k}(z) \) is an automorphic form with respect to \( \text{Sl}_2(\mathbb{Z}) \) of weight \( 2k \).

Exercise 17. Let \( f \neq 0 \) be a modular form of weight \( k \). For \( x \in \mathbb{H}^* \) let \( v_x(f) \) denote the order of \( f \), i.e., the integer \( m \) such that \( (z-x)^{-m}f(z) \) is holomorphic and non-zero at \( z = x \). Let \( \Gamma = \text{Sl}_2(\mathbb{Z}) \) and denote by \( e_x \) the order of the stabiliser of \( x \in \mathbb{H} \) in \( \Gamma / \{ \pm 1 \} \). Show that

\[ v_{\infty}(f) + \sum_{x \in \Gamma \setminus \mathbb{H}} \frac{1}{e_x} v_x(f) = \frac{k}{12}. \]

(Hint: Use Cauchy’s Theorem to \( \frac{df}{dz} \).)

Exercise 18. Use the previous exercises to compute the dimension of the space of modular forms of weight \( k \) for each \( k < 12 \).