Exercise 1.  (i) Let $\Gamma \subseteq \text{SL}_2(\mathbb{R})$ be a discrete subgroup. A point $z \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial \overline{\mathbb{H}}$ is called \textit{parabolic for} $\Gamma$ if it is fixed by some parabolic element in $\Gamma$. Prove that the action of $\Gamma$ preserves the set of parabolic points for $\Gamma$.

(ii) What is the set of parabolic points for $\Gamma = \text{SL}_2(\mathbb{Z})$? How many orbits does $\text{SL}_2(\mathbb{Z})$ have on this set?

(iii) If $G$ is a group and $H_1, H_2 \subseteq G$ are subgroups, then $H_1$ and $H_2$ are called \textit{commensurable} if $[H_1 : H_1 \cap H_2] < \infty$ and $[H_2 : H_1 \cap H_2] < \infty$. Show that if $\Gamma, \Gamma' \subseteq \text{SL}_2(\mathbb{R})$ are commensurable discrete subgroups, then the set of parabolic points for $\Gamma$ is the same as the set of parabolic points for $\Gamma'$.

(iv) Prove that $\Gamma = \text{SL}_2(\mathbb{Z})$ and $\Gamma' = g^{-1} \Gamma g$ are commensurable for every $g \in \text{GL}_2(\mathbb{Q})$.

Exercise 2. Suppose $\gamma_1, \gamma_2 \in \text{SL}_2(\mathbb{Z})$, $\gamma_1, \gamma_2 \neq \pm \text{id}$. Show that if $\gamma_1$ and $\gamma_2$ commute, they are of the same type, i.e. they are both elliptic, hyperbolic or parabolic.

Exercise 3. Let $\Gamma \subseteq \text{SL}_2(\mathbb{R})$ be a discrete subgroup. Show that $\Gamma$ is finite if and only if every $\gamma \in \Gamma$, $\gamma \neq \pm \text{id}$, is elliptic.

(Hint: For “$\Rightarrow$” first show that all $\gamma \in \Gamma$ must have the same fixed point.)

*Exercise 1.* Use the fundamental domain $D$ for $\text{SL}_2(\mathbb{Z})$ from Exercise 0.2 to compute the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ for $d \in \{-4, -3, -7\}$. 