Exercise 16. Show that for all $X > 0$, 

$$\sum_{1 \leq n \leq X} \frac{1}{n} = \log X + \gamma + O\left(\frac{1}{X}\right),$$

where $\gamma$ is Euler’s constant, and the implied constant of the error term is independent of $X$.

Exercise 17. Show that for all $r > 0$,

(i) 

$$\int_0^r \frac{\log t}{\sqrt{t(r-t)}} \, dt = \pi \log r - 2\pi \log 2.$$

(Hint: Reduce to the case $r = 1$.)

(ii) 

$$-ir \int_0^\infty e^{iru} \log(1 - e^{-u}) \, du = -\frac{\Gamma'(1 - ir)}{\Gamma(1 - ir)} - \gamma.$$

(Hint: Use that the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ can be written as $\psi(x + 1) = -\gamma + \int_0^1 \frac{1-e^t}{1-t} \, dt$ for $\Re x > 0$.)