Exercise 29. Let \( f \) be a Maass cusp form with respect to \( \text{Sl}_2(\mathbb{Z}) \). Prove that \( f \in L^2(\text{Sl}_2(\mathbb{Z}) \backslash \mathcal{H}). \)

Exercise 30. (i) Let \( k \geq 0 \) be an even integer. Let \( f \) be a (holomorphic) cusp form of weight \( k \) for \( \text{Sl}_2(\mathbb{Z}) \) and let \( f(z) = \sum_{n>0} a_n e^{2\pi i nz} \) be its Fourier expansion. Show that there exists a constant \( C > 0 \) such that \( |a_n| \leq Cn^{k/2} \) for all \( n \).

(Hint: Integrate \( e^{-2\pi i nx} f(x + iy) \) over \( x \in [0,1] \).)

(ii) Let \( f \) be a Maass cusp form for \( \text{Sl}_2(\mathbb{Z}) \) with Laplace eigenvalue \( s(s - 1) \) and let \( f(z) = \sum_{n \neq 0} a_n \sqrt{\gamma} K_{s - \frac{1}{2}}(2\pi |n|y) e^{2\pi i nz} \) be its Fourier expansion. Prove that there exists a constant \( C > 0 \) such that \( |a_n| \leq Cn^{1/2} \) for all \( n \).

Exercise 31. For \( k \in \mathbb{Z}, k \geq 2 \), let \( G_{2k}(z) = \sum_{(m,n) \neq (0,0)} (mz + n)^{-2k} \). Show that:

(i)

\[
G_{2k}(z) = 2\zeta(2k) \sum_{\gamma \in B \backslash \text{Sl}_2(\mathbb{Z})} \left( \frac{d(\gamma z)}{dz} \right)^k
\]

where \( \zeta \) is the Riemann zeta function and \( B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}_2(\mathbb{Z}) \mid c = 0 \right\} \).

(ii) The Fourier expansion of \( G_{2k} \) is

\[
G_{2k}(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k - 1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i nz},
\]

where \( \sigma_k(n) = \sum_{m \in \mathbb{N}, m|n} m^k \).

(Hint: Use the cotangent identity \( \pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} + \frac{1}{z-n} \right) \).

(iii) \( G_{2k}(z) \) is an automorphic form with respect to \( \text{Sl}_2(\mathbb{Z}) \) of weight \( 2k \).

Exercise 32. Let \( G = \text{Sl}_2(\mathbb{R}) \) and \( K = \text{SO}(2) \). Let \( \mathcal{S} \) denote the set of positive definite symmetric matrices in \( G \).

(i) Show that the maps \( G/K \rightarrow \mathcal{S}, K ightarrow gK \mapsto gg^t \), and \( \mathcal{S} \rightarrow \mathcal{H}, A \mapsto z \in \mathcal{H} \) for \( z \) the unique point with \( (z \ 1)A(\frac{1}{z}) = 0 \) are bijections and preserve the action of \( G \). Further show that the inverse of the last map is given by \( z = x + iy \mapsto y^{-1} \left( \frac{1}{-x} z^2, -x, y^2 \right) =: Az \).

(ii) For \( s \in \mathbb{C} \) with \( \Re s > 1 \) and \( Y \in \mathcal{S} \) define \( Z(Y, s) = \frac{1}{2} \sum_{v \in \mathbb{Z}^2 \backslash \{0\}} (v^t Y v)^{-s} \), where \( v^t \) denotes the transpose of \( v \). Prove that

\[
\zeta(2s)E_\infty(z, s) = Z(A_z, s),
\]

where \( E_\infty(z, s) \) is the (non-analytic) Eisenstein series introduced in the lectures.