Exercise 25.  
(i) Recall that the Legendre function can be defined as \( P_\nu(z) = F(-\nu, \nu + 1, 1; \frac{1-z}{2}) \) for \( F \) the hypergeometric function. Show that if \( \nu \) is fixed with \( \Re \nu > -\frac{1}{2} \) and \( \Re z > 0 \), then 
\[
P_\nu(z) \sim \frac{(2z)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu + 1)}
\] 
as \( z \to \infty \).
(ii) Define the \( K \)-Bessel function \( K_\nu(z) \) for \( \Re z > 0 \) by 
\[
K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}(u^2+1)} u^{\nu-1} du.
\]
Prove that if \( \nu \) with \( \Re \nu > 0 \) is fixed and \( \Re z > 0 \), then 
\[
K_\nu(z) \sim 2^{\nu-1} \Gamma(\nu) z^{-\nu}
\] 
as \( z \to 0 \).
(iii) Let \( r, s \) be real numbers. Show that for \( \Re \nu > 0 \), 
\[
\int_{-\infty}^{\infty} (s^2 + u^2)^{-\nu} e^{2iu} du = \begin{cases} 
\frac{2^{\nu-1} \pi^{\nu-\frac{1}{2}} \Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)} |r|^\nu K_{\nu-\frac{1}{2}}(2|rs|) & \text{if } r, s \neq 0, \\
|s|^{1-2\nu} \Gamma(\frac{\nu}{2}) \Gamma(\nu-\frac{1}{2}) & \text{if } r = 0, s \neq 0.
\end{cases}
\] 
(Hint: Multiply by \( \Gamma(\nu) \) and use \( \Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt \).)

Exercise 26. Let \( G = \text{Sl}_2(\mathbb{R}) \), \( K = \text{SO}(2) \), and let \( f \in C_c^\infty(K \backslash G/K) \). Suppose \( \varphi \) is an eigenfunction for \( \Delta \) with eigenvalue \( s(s-1) \). Show that for every \( g \in G \), we have 
\[
\varphi \ast f(g) = \varphi(g) \int_{\mathcal{H}} f(x + iy) y^{-s-1} dx dy.
\]

Exercise 27. Let \( G \) be a group, \( H \) a Hilbert space, and \( \pi : G \to \text{End}(H) \) a group action of \( G \) on \( H \) such that for every \( g \in G \), the endomorphism \( \pi(g) \) is unitary. Moreover, suppose that for every fixed \( h \in H \), the map \( G \ni g \mapsto \pi(g)h \in H \) is continuous. Show that \( \pi \) is a continuous representation of \( G \), i.e., show that \( G \times H \ni (g, h) \mapsto \pi(g)h \in H \) is continuous.

Exercise 28. Let \( G = \text{Sl}_3(\mathbb{R}) \), \( K = \text{SO}(2) \), and let \( \chi : K \to \mathbb{C}^\times \) be a homomorphism. Let \( C_c^\infty(G, K, \chi) \) consist of all functions \( f \in C_c^\infty(G) \) such that \( f(k_1 g k_2) = \chi(k_1) f(g) \chi(k_2) \) for all \( k_1, k_2 \in K \), \( g \in G \). Show that \( C_c^\infty(G, K, \chi) \) is a commutative algebra under convolution.