Exercise 37. Let $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ be a discrete subgroup such that $\Gamma_\infty = \{ (\pm 1, n) \mid n \in \mathbb{Z} \} \subseteq \Gamma$. Show that we have a disjoint double coset decomposition

$$\Gamma_\infty \backslash \Gamma / \Gamma_\infty = \Gamma_\infty \cup \bigcup_{c \geq 1} \bigcup_{d \mod c} \Gamma_\infty \left( \begin{smallmatrix} c & * \\ d & c \end{smallmatrix} \right) \Gamma_\infty,$$

where the sum runs over all $c, d$ such that there exist $a, b$ with $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$, and $\left( \begin{smallmatrix} c & * \\ d & c \end{smallmatrix} \right)$ denotes a representative of this set of matrices.

Exercise 38. Define for $z \in \mathcal{H}$

$$G_2(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z} : n \neq 0 \text{ if } m = 0} \frac{1}{(mz + n)^2}.$$

(i) Show that for every integer $m \geq 1$ we have $\sum_{n \in \mathbb{Z}} (mz + n)^{-2} = -(2\pi)^2 \sum_{d \geq 1} d e^{2\pi i m \zeta}$.

(ii) Show that $E_2(z) := \frac{1}{\zeta(2)} G_2(z) = 1 - 24 \sum_{d \geq 1} \sigma_1(d) e^{2\pi i dz}$ and this sum converges absolutely.

(iii) Prove that $E_2(-1/z) = z^2 E_2(z) + \frac{12}{2\pi i z}$.

(Hint: Consider the double sum over $c_{mn} = \frac{1}{(mz+n)^2}$ for $c_{mn} = \frac{1}{mz+n-1} - \frac{1}{mz+n}$)

Exercise 39. Suppose $n > 0$ is an integer which is divisible by 8 and let $A \in \text{SL}_n(\mathbb{Z})$ be a positive definite symmetric matrix satisfying $v^t A v \equiv 0 \pmod{2}$ for all $v \in \mathbb{Z}^n$.

(i) Use the Poisson summation formula to show that $\theta(A, z)$ is a holomorphic automorphic form of weight $n/2$ with respect to $\text{SL}_2(\mathbb{Z})$, where

$$\theta(A, z) := \sum_{v \in \mathbb{Z}^n} e^{\pi i (v^t A v) z}.$$

(ii) Compute $c_0$ such that $\theta(A, z) - c_0 G_{n/2}(z)$ is a cusp form.

(iii) For a non-negative integer $m$ let

$$r_A(m) = \left| \{ v \in \mathbb{Z}^n \mid v^t A v = 2m \} \right|.$$

Show that there is a constant $C > 0$ such that

$$r_A(m) \leq \frac{(2\pi)^{n/2}}{\zeta(n/2)(n/2-1)!} \sigma_{n/2-1}(m) \leq C m^{n/4}$$

for all $m \geq 1$. 

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Exercise 40. Let $E_{\infty}(z,s)$ denote the usual non-analytic Eisenstein series for $\text{SL}_2(\mathbb{Z})$. Let $f : (0, \infty) \rightarrow \mathbb{C}$ be smooth and compactly supported. Define

$$E_f(z) := \frac{1}{4\pi} \int_0^{\infty} f(t) E_{\infty}(z, \frac{1}{2} + it) dt.$$ 

Show that there is a constant $c > 0$ such that $|E_f(z)| \leq c \frac{\sqrt{y}}{\log y}$ as $y \to \infty$, and conclude that $E_f \in L^2(\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H})$. 