MAPPING SURGERY TO ANALYSIS

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Abstract. In a series of three talks we present a construction, due to Higson and Roe, of a map from the geometric surgery exact sequence to the analytic surgery exact sequence.

Surgery.

Let \( X \) be a closed \( n \)-dimensional CAT-manifold, where CAT = DIFF, PL or TOP. A basic goal in topology is to understand the structure set \( S^{\text{CAT}}(X) \) of \( X \), which is a pointed set whose elements are represented by homotopy equivalences \( f : M \to X \), where \( M \) is some closed \( n \)-dimensional manifold, modulo the relation of CAT-isomorphism in the source. This means that \( f_0 : M_0 \to X \) is equivalent to \( f_1 : M_1 \to X \) if there exists a CAT-isomorphism \( h : M_0 \to M_1 \) such that \( f_1 \circ h \simeq f_0 \). The CAT-isomorphism means a diffeomorphism if CAT = DIFF, a PL-homeomorphism if CAT = PL or a homeomorphism if CAT = TOP.

The structure set \( S^{\text{DIFF}}(S^n) \) is the group of exotic spheres in dimension \( n \).

The Borel conjecture predicts that the structure set \( S^{\text{TOP}}(X) = 0 \) if \( X \) is a closed aspherical manifold of dimension \( \geq 5 \).

The basic tool to study \( S(X) \), when \( \dim(X) \geq 5 \) is the geometric surgery exact sequence (GSES):

\[
\cdots \to N_{\beta}(X \times I) \xrightarrow{\partial} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\partial} S(X) \xrightarrow{\eta} N(X) \xrightarrow{\partial} L_n(\mathbb{Z}\pi),
\]

where \( \pi = \pi_1(X) \). It works for all settings of CAT, so that notation is dropped.

In (GSES), the symbol \( N(X) \) denotes the normal invariants of \( X \), which is a (bordism type) generalized (co-)homology theory. Hence, standard tools of algebraic topology can usually be applied to calculate it, at least in principle. An element in \( N(X) \) is represented by a degree one normal map \((f,b) : M \to X\) from some closed \( n \)-dimensional manifold \( M \) to \( X \). The terminology means just that there is a map \( f \) of manifolds of degree one in the usual sense, together with some bundle data, which are hidden behind the symbol \( b \).

The symbol \( L_n(\mathbb{Z}\pi) \) denotes the so-called \( L \)-group of the group ring \( \mathbb{Z}\pi \) which only depends on \( \pi \) and \( n \) modulo 4. If \( n = 2k \) is even then it is defined as the Witt-group of \((-1)^k\)-quadratic forms over \( \mathbb{Z}\pi \), if \( n = 2k+1 \) is odd then it is defined in terms of automorphisms of \((-1)^k\)-quadratic forms over \( \mathbb{Z}\pi \). Alternatively, there is a unified definition for any \( n \) as the cobordism group of \( n \)-dimensional quadratic Poincaré chain complexes over \( \mathbb{Z}\pi \), which is a definition in terms of algebraic theory of surgery of Ranicki. In any case, the \( L \)-group is something algebraic.
The map $\theta : N(X) \to L_n(\mathbb{Z}\pi)$ is the surgery obstruction map. It has the property that, when $n \geq 5$ then $\theta(f, b) = 0$ if and only if $(f, b)$ is normally cobordant (meaning cobordant together with certain bundle data) to a degree one map $(f', b')$ where $f'$ is a homotopy equivalence.

For example in the simply connected case when $n = 4k$, the $L$-group is $L_{4k}(\mathbb{Z}) = \mathbb{Z}$ and the surgery obstruction map is given by

$$\theta : ((f, b) : M \to X) \mapsto (\text{Sign}(M) - \text{Sign}(X))/8.$$ 

In general one can think of elements of $L_n(\mathbb{Z}\pi)$ as generalized signatures of quadratic forms over $\mathbb{Z}\pi$ which refine the symmetric bilinear form obtained as the difference between the symmetric bilinear forms in the middle homology of $M$ and $X$.

More information can be found in [Lüc02, chapters 3-5]. If you have never seen this before you may want to consult the Wikipedia article [Wik] about this topic as a first step.

Note that the Borel Conjecture can be translated into a statement about the surgery obstruction map in the topological category. The closely related Novikov Conjecture, which was originally stated in term of higher signatures, can be rephrased into saying that the surgery obstruction map is rationally injective, when $X = B\pi$. This statement is independent of the category, because the sequences for $\text{CAT} = \text{DIFF}$ and $\text{CAT} = \text{TOP}$ are rationally equal.

**Analysis.**

It is in general difficult to understand the geometric surgery exact sequence for a given manifold $X$. Although the normal invariants are tractable, the $L$-groups are hard to compute and the surgery obstruction map is even harder to determine. The methods for attacking these problems depend heavily on whether $\pi$ is finite or infinite. For $\pi$ finite a great deal is known [HT00].

For $\pi$ infinite there are various approaches. In view of the Borel Conjecture one way to attack the geometric surgery exact sequence is via controlled topology.

The aim of this series of talks is to present another possible attack on the (GSES) when $\pi$ is infinite, namely via analysis. The analysis comes into play naturally because when $\pi$ is infinite, then the group ring $\mathbb{Z}\pi$ is very difficult to understand from the ring theory point of view, but the reduced $C^*$-algebra $C^*_r(\pi)$, which is a certain completion of $\mathbb{C}\pi = \mathbb{C} \otimes \mathbb{Z}\pi$, can and has been successfully studied via analysis, more concretely via its $K$-theory. A good, and more detailed, explanation of why and how analysis arises in this context can be found in [Ros95].

More specifically the relationship of the (GSES) to analysis is via another exact sequence, called the analytic surgery exact sequence (ASES), and a map from (GSES) to (ASES), as presented in [HR05a], [HR05b], [HR05c].

The analytic surgery exact sequence (ASES) has the following shape:

\[
(0.2) \quad \cdots \to K_{n+1}(X) \to K_{n+1}(C^*_r\pi) \to K_{n+1}(D_\pi^*X) \to K_n(X) \to K_n(C^*_r\pi),
\]

Modulo certain technical issues this exact sequence is obtained as the 6-term exact sequence in the $K$-theory of $C^*$-algebras. Without going into details now, we mention that the respective terms can be described in terms of the following concepts:
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Our goal in the three talks will be to sketch the construction of the commutative diagram (where CAT = DIFF and up to factors of 2):

\[
\cdots \xrightarrow{\beta} N_0(X \times I) \xrightarrow{S(X)} N(X) \xrightarrow{\alpha} L_n(\mathbb{Z}) \xrightarrow{\beta}\cdots
\]

\[
\cdots K_{n+1}(X) \xrightarrow{\beta} K_{n+1}(C_r^*\pi) \xrightarrow{\alpha} K_n(D^*_\pi X) \xrightarrow{\beta} K_n(X) \xrightarrow{\gamma} K_n(C_r^*\pi).
\]

The central concept used in the definition of the maps \(\alpha, \beta, \gamma\) is that of a Hilbert-Poincaré complex of \(X\)-modules. This will be a chain complex of \(X\)-modules equipped with certain operators. On one hand it will be shown that such chain complexes have “signatures”, that means they determine classes in the \(K\)-theory groups of the respective \(C^*\)-algebras. On the other hand it will be shown that they arise in the geometric situations corresponding to the top row of the diagram. Making the above more precise is going to be the main content of our talks.

It should be noted that the definition of the (ASES) was strongly inspired by the controlled surgery exact sequence (CSES). This is an exact sequence obtained by applying \(L\)-theory to certain additive categories with duality associated to \(X\). For a simplified version of the analogy in algebraic \(K\)-theory the reader may consult the user’s guide paper [Ros]. By the algebraic \(K\)-theory analogue of the (CSES) we mean the exact sequence obtained by applying algebraic \(K\)-theory to the first row of the diagram on page 12 of that paper. In particular, finite propagation locally compact operators are analogous to conditions on objects and morphisms of the category \(\mathcal{D}_\pi(X \times [0,1]; \mathcal{A})\), finite propagation pseudolocal operators are analogous to the conditions of the category \(\mathcal{D}(X \times [0,1]; \mathcal{A})\), i.e. continuous control at infinity, and finally, Paschke duality is analogous to the property that the algebraic \(K\)-theory of the corresponding germ category is a homology theory.

Applications.

So far the map (GSES) to (ASES) could only be used to re-prove results already known. Nevertheless, the new proofs thus obtained have a virtue of being more conceptual in some sense. Here we only hint at how this goes.

- Novikov Conjecture: Recall that one formulation of the Novikov Conjecture says that the surgery obstruction map is rationally injective for \(X = B\pi\). If we can show that the map \(\beta\) and the index map \(K_n(X) \to K_n(C_r^*(\pi))\)
are rationally injective, then we get from the commutativity that the map \( \theta \) must be rationally injective.

- \( \rho \)-invariants: A result of Keswani says that if the Baum-Connes conjecture is true for the group \( \pi \) then the \( \rho \)-invariant \( \rho_\pi(M) \in \mathbb{R} \) is a homotopy invariant. This can be phrased as saying that the relative \( \rho \)-invariant is zero as a map from the structure set to \( \mathbb{R} \). The proof of Higson and Roe in [HR10] says that this is so essentially because the \( \rho \)-invariant factors through the term \( K_{n+1}(D^*(X)) \) which is zero if the Baum-Connes conjecture holds. (Note for the experts: this result is proved in the setting where \( C^*_r(\pi) \) is replaced by \( C^*_\text{max}(\pi) \).)

Further interest is in the above theory because of the relation to the so-called positive scalar curvature exact sequence (PSCES) of Stolz. Briefly speaking here one replaces the “signature operators” by the Dirac operators for spin manifolds.

The topological case. Because of the already mentioned analogy with the (CSES) it should be hoped that if one replaces the (ASES) with the real \( C^* \)-algebras (Higson and Roe work with complex \( C^* \)-algebras all the time) then it should be possible to obtain a map from the TOP-(GSES) to the real-case (ASES) which localized away from 2 should conjecturally be an isomorphism. This “topological problem” is mentioned explicitly in the last remark of [HR05c], as a non-trivial, but desirable, task.

Schedule.

Talk 1. (Ján Špakula: Analytic surgery exact sequence)
Introduction. \( C^* \)-algebras, functional calculus, Hilbert modules, unbounded operators, \( K \)-theory of \( C^* \)-algebras, 6-term exact sequence in the \( K \)-theory of \( C^* \)-algebras, \( K \)-homology, analytic surgery exact sequence.

Talk 2. (Philipp Kühl, Wolfgang Steimle: Analytic signatures)
Hilbert-Poincaré (HP) complexes, their signatures in \( K \)-theory and their properties, examples of obtaining HP-complexes from various geometric situations.

Talk 3. (Tibor Macko: Construction of the map (GSES) to (ASES))
Review of (GSES), construction of the maps \( \alpha, \beta, \gamma \), discussion of the commutativity, applications and concluding remarks.

References

