Singular Riemannian foliations on spaces without conjugate points

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We describe the topological structure of cocompact singular Riemannian foliations on Riemannian manifolds without conjugate points. We prove that such foliations are regular and developable and have regular closures. We deduce that in some cases such foliations do not exist.

Keywords: Negative curvature, focal points, geodesic flow

1. Introduction

Riemannian manifolds of non-negative curvature often admit large groups of isometries. Moreover, there are many famous examples of Riemannian foliations on such spaces, like the Hopf fibrations and of singular Riemannian foliations, such as isoparametric foliations. Singular Riemannian foliations on non-negatively curved manifolds tend to be homogeneous and seem to be rather rigid objects. On the other hand, (singular) Riemannian foliations on such spaces are often related to other rigidity questions (cf.[1–8]).

If one changes the sign of the curvature then the situation seems to be completely different on the first glance. For instance, in a simply connected negatively curved manifold there are infinite-dimensional families of Riemannian submersions to the real line and there seem to be no hope of getting any kind of control of such objects. However, for *compact* manifold of non-positive curvature the situation seems again be very similar to the "rigid" non-negatively curved world. The first indication is the famous result of Bochner ([9]) that describes connected isometry groups of such spaces. In particular, the isometry group of a compact negatively curved manifold turns out to be finite. Indeed, this is the case for any Riemannian metric on such manifolds, since they have positive minimal volume ([10]). In[11] it is shown that the existence of a Riemannian flow on a compact manifold forces its minimal volume to be zero, thus Riemannian flows do not exist on compact negatively curved manifolds. Finally, A. Zeghib proved in[12] Theorem F that on a compact negatively curved manifold there are no (regular) Riemannian foliations at all.

Remark 1.1. Previously, the non-existence of regular Riemannian foliations on compact negatively curved manifolds was claimed in[13] and, in special cases in[14] and.[15] However, these proofs are not correct, cf.[16] and the discussion in,[12] pp.1435-1436.

Here, we generalize the non-existence theorem to singular Riemannian foliations, a broad generalization of regular Riemannian foliations and isometric group actions. We prove:

Theorem 1.1. *Singular Riemannian foliations do not exist on compact negatively curved manifolds.*

Remark 1.2. In[17] the non-existence result was proved under the assumption that the singular Riemannian foliations has horizontal sections, i.e., that the horizontal distribution in the regular part is integrable.

In fact, in analogy with[17] we prove in a broader context that a singular Riemannian foliation on a compact negatively curved manifold cannot have singular leaves, i.e., it must be a regular Riemannian foliation. Then we apply.[12] Our main result result used in Table 1.1 describes the topology of singular Riemannian foliations in the following more general situation.

Theorem 1.2. *Let M be a complete Riemannian manifold without conjugate points and let $\mathcal{F}$ be a singular Riemannian foliation on M such that the space of leaves $M/\mathcal{F}$ has bounded diameter with respect to the quotient pseudo-metric. Then $\mathcal{F}$ is a regular foliations and has a regular closure $\bar{\mathcal{F}}$. The quotient $M/\bar{\mathcal{F}}$ is a good Riemannian orbifold without conjugate points. The leaves of the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering $\tilde{M}$ of M are closed and contractible. They are given by a Riemannian submersion $p : \tilde{M} \to B$ to a contractible manifold B.*

In the case of a simply connected total space M we deduce from the last part of Table 1.2:

Corollary 1.3. *Let M be a complete, simply connected Riemannian manifold without conjugate points. Then there are no non-trivial singular Riemannian foliations $\mathcal{F}$ on M with a bounded quotient $M/\mathcal{F}$.*

Remark 1.3. In the case $M = \mathbb{R}^n$ the last result was recently shown in[18] using different methods.

The proof of Table 1.2 is divided into a geometric and a topological part. In the geometric part, similar to[17] , we analyze the structure of $\bar{\mathcal{F}}$ and prove that regular leaves of $\bar{\mathcal{F}}$ do not have focal points (this already implies the first two claims in our theorem). The idea of the proof is that focal points of regular leaves correspond either to crossings of singular leaves or to conjugate points in the quotient. Now, the Poincare recurrence theorem for the quasi-geodesic flow on the quotient $M/\bar{\mathcal{F}}$ (cf.[19] Theorem 1.6; here we use the compactness of the quotient) tells us that the existence of a single focal point would imply the existence of a horizontal geodesic with arbitrary many focal points. (This is a modified form of the statement that on a compact Riemannian manifold with uniformly bounded number of conjugate points along all geodesics, there are no conjugate points at all). However, the absence of conjugate points on M implies that each leaf has at most $\dim(M)$ focal points along any horizontal geodesic. This contradiction finishes the geometric part of the proof.

The remaining part of the proof is finished by using the following purely topological observation.

Prop 1.1. Let M be an aspherical manifold with a complete Riemannian metric. Let $\mathcal{F}$ be a Riemannian foliation on M with dense leaves. Then the leaves of the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering $\tilde{M}$ are closed and contractible. The lifted foliation $\tilde{\mathcal{F}}$ is given by a Riemannian submersion $p : \tilde{M} \to B$ onto a contractible homogeneous manifold B.

2. Preliminaries

A *transnormal system* $\mathcal{F}$ on a Riemannian manifold M is a decomposition of M into smooth, injectively immersed, connected submanifolds, called leaves, such that geodesics emanating perpendicularly to one leaf stay perpendicularly to all leaves. A transnormal system $\mathcal{F}$ is called a *singular Riemannian foliation* if there are smooth vector fields X_i on M such that for each point $p \in M$ the tangent space $T_pL(p)$ of the leaf $L(p)$ through p is given as the span of the vectors $X_i(p) \in T_pM$. We refer to[8,19,20] for more on singular Riemannian foliations. Examples of singular Riemannian foliations are (regular) Riemannian foliations and the orbit decomposition of an isometric group action.

If M is complete then leaves of a transnormal system $\mathcal{F}$ are equidistant and the distance between leaves define a natural pseudo-metric on the space of leaves. This pseudo-metric space is bounded if and only if some finite tubular neighborhood of a leaf coincides with the whole space. If $\mathcal{F}$ is *closed*, i.e., if leaves of $\mathcal{F}$ are closed then the quotient $B = M/\mathcal{F}$ is a complete, locally compact, geodesic metric space, that is compact if and only if it is bounded. Moreover, B is an Alexandrov space with curvature locally bounded below. If it is compact, its Hausdorff measure is finite.

Let $\mathcal{F}$ be a singular Riemannian foliation on the Riemannian manifold M. The *dimension of* $\mathcal{F}$, $\dim(\mathcal{F})$, is the maximal dimension of its leaves. For $s \leq \dim(\mathcal{F})$ denote by Σ_s the subset of all points $x \in M$ with $\dim(L(x)) = s$. Then Σ_s is an embedded submanifold of M and the restriction of $\mathcal{F}$ to Σ_s is a Riemannian foliation. For a point $x \in M$, we denote by Σ^x the connected component of Σ_s through x, where $s = \dim(L(x))$. We call the decomposition of M into the manifolds Σ^x the *canonical stratification* of M. The subset $\Sigma_{\dim(\mathcal{F})}$ is open, dense and connected in M. It is the *regular stratum* M. It will be denoted by M_0 and will also be called the set or regular points of M. All other strata Σ^x are called *singular strata*.

Let $\mathcal{F}$ be a singular Riemannian foliation on a complete Riemannian manifold M. Then the decomposition $\bar{\mathcal{F}}$ of M into closures of leaves of $\mathcal{F}$ is a transnormal system, that we will call the closure of $\mathcal{F}$. The restriction of $\bar{\mathcal{F}}$ to each stratum Σ of M (with respect to $\mathcal{F}$) is a singular Riemannian foliation.

For a transnormal system $\mathcal{F}$ on M, we will call a point $x \in M$ regular if its leaf is regular, i.e., if it has the maximal dimension. The closure of a singular leaf of a singular Riemannian manifold $\mathcal{F}$ on a complete Riemannian manifold M is a singular leaf of $\bar{\mathcal{F}}$. In particular, if $\bar{\mathcal{F}}$ does not have singular leaves then $\mathcal{F}$ is a (regular) Riemannian foliation.

Let $M, \mathcal{F}, \bar{\mathcal{F}}$ be as above. Then M gets a canonical stratification with respect to $\bar{\mathcal{F}}$ that is finer than the canonical stratification with respect to $\mathcal{F}$, such that the restriction of $\bar{\mathcal{F}}$ to each stratum is a Riemannian foliation. The main stratum M_0 is again open and dense. This defines a canonical stratification of the quotient $B = M/\bar{\mathcal{F}}$ into smooth Riemannian orbifolds. The main stratum M_0 is projected to the main stratum B_0 of B that is open and dense in B. If B is compact, the orbifold B_0 has finite volume.

Horizontal geodesics of the transnormal system $\bar{\mathcal{F}}$ are projected to concatenations of geodesics in B. Each horizontal geodesic in the regular part M_0 is projected to an orbifold-geodesic in B_0. Let γ_1 and γ_2 be horizontal geodesics whose projections η_1 and η_2 to B coincide initially. Then η_1 and η_2 coincide on the whole real line (cf.[19] and,[21] for the case of singular Riemannian foliation and[22] and[18] for the case of closed transnormal systems).

Therefore, the geodesic flow on M restricted to the space of horizontal vectors projects to a "quasi-geodesic" flow on the "unit tangent bundle" of B. Note, finally, that for each regular leaf L of $\bar{\mathcal{F}}$ and each horizontal geodesic γ starting on L, each intersection point of γ with a singular leaf is a focal point of L along γ.

We finish this section with an easy application of the Poincaré's recurrence theorem:

Lemma 2.1. *Let B_0 be a (non-necessarily) complete Riemannian orbifold with finite volume. Let V be a non-empty open subset of the unit tangent bundle U_0 of B_0. Assume that the geodesic flow $\phi_t(v)$ is defined for all $v \in V$ and all $t > 0$. Let a positive real number T be given. Then there is a non-empty open subset $V_0 \subset V$ and $\bar{T} > T$ such that $\phi_{\bar{T}}(V_0) \subset V$, and such that $\phi_t(v)$ is defined for all $v \in V_0$ and all $t \in [-T, 0]$.*

3. Geometric arguments

Using the preparation from the last section, we can now easily prove the geometric part of Table 1.2.

Let $M, \mathcal{F}$ be as in Table 1.2. Consider the closure $\bar{\mathcal{F}}$ of $\mathcal{F}$. Let B denote the compact quotient $B = M/\bar{\mathcal{F}}$ with the projection $q : M \to B$. Let B_0 be the regular part of B, i.e., the set of all regular leaves of $\bar{\mathcal{F}}$ in M.

We are going to prove that all regular leaves of $\bar{\mathcal{F}}$ have no focal points in M. Assume the contrary. Denote by M_0 the regular part of M (with respect to $\bar{\mathcal{F}}$; the original singular foliation $\mathcal{F}$ will not be used in this section). Let $\mathcal{H}$ be the horizontal distribution on M_0. Let $\mathcal{H}^1$ be the space of unit vectors in $\mathcal{H}$, with the foot point projection $p : \mathcal{H}^1 \to M$. For $h \in \mathcal{H}^1$ let $\gamma^h : [0, \infty) \to M$ denote the horizontal geodesic starting in the direction of h. By $L(h)$ we denote the leaf of $\bar{\mathcal{F}}$ through the foot point $p(h) \in M$. By $f(h)$ we will denote the $L(h)$-index of γ^h, i.e., the number of $L(h)$-focal points along γ^h. By Λ^h we denote the Lagrangian space of normal Jacobi fields along γ^h that consists of $L(h)$-Jacobi fields (cf.[23]). As in,[23] we denote for an interval $I \subset (0, \infty)$ by $\mathrm{ind}_{\Lambda^h}(I)$ the number of $L(h)$-focal points along γ^h in $\gamma^h(I)$.

Since there are no conjugate points in the manifold M, the function f is bounded by $\dim(M)$ on $\mathcal{H}^1$ (,[23] Corollary 1.2). Let m be the maximum of the function f, that is positive by our assumption. Choose some $h_0 \in \mathcal{H}^1$ with $f(h_0) = m$. Choose some $T > 0$ such that all (precisely m, when counted with multiplicity) $L(h_0)$-focal points along γ^{h_0} come before T, i.e., $\mathrm{ind}_{\Lambda^{h_0}}((0, T)) = m$. By continuity of indices and maximality of m, we find an open neighborhood V of h_0 in $\mathcal{H}^1$, with $\mathrm{ind}_{\Lambda^h}((0,T)) = m$, for all $h \in V$.

Since each intersection of γ^h with a singular leaf happens in a focal point, for all $h \in V$, the geodesic $\gamma^h : [T, \infty) \to M$ does not intersect singular leaves. Thus, $\gamma^h([T,\infty))$ is contained in M_0 and, for its projection $\eta^h = q \circ \gamma^h$, we have $\eta^h([T,\infty) \subset B_0$. Due to Lemma 2.1, we find an open subset V_0 of V and some $\bar{T} > T$ such that for all $h \in V_0$ we have $\gamma_h[0,\infty) \subset M_0$ and $(\gamma^h)'(\bar{T}) \in V_0$.

Choose now some $h \in V_0$. Since γ^h is contained in M_0, the projection $\eta^h = q \circ \gamma^h$ is an orbifold-geodesic in B_0. Moreover, $L(h)$-focal points along γ^h correspond to conjugate points along η^h. For the Jacobi equation along η^h (in terms of,[23] this is the transversal Jacobi equation introduced in[8]), we have the following picture. The point $\eta^h(\bar{T})$ has at least one conjugate point along η^h in the interval $(\bar{T}, \bar{T}+T)$ (in fact, there are precisely m such points counted with multiplicities). Therefore, $\eta^h(0)$ has at least one conjugate point along η^h in the interval $(\bar{T}, \bar{T}+T)$ (,[23] Corollary 1.3). Since $\bar{T} > T$, by assumption, we get an $L(h)$-focal point $\gamma^h(t)$ along γ^h for some $t > T$, in contradiction to $\mathrm{ind}_{\Lambda^h}(0,\infty) = \mathrm{ind}_{\Lambda^h}(0,T)$.

Thus, we have proved, that all regular leaves of $\bar{\mathcal{F}}$ have no focal points. Hence $\bar{\mathcal{F}}$ has no singular leaves. Therefore, $\mathcal{F}$ and $\bar{\mathcal{F}}$ are regular Riemannian foliation. Moreover, since focal points of leaves of a closed regular Riemannian foliation correspond to conjugate points in the quotient orbifold, we deduce that the quotient $B = M/\bar{\mathcal{F}}$ has no conjugate points.

4. Topological arguments

First, we are going to prove Proposition 1.1. Thus let M be an aspherical manifold with a complete Riemannian metric. Let $\mathcal{F}$ be a Riemannian foliation on M with dense leaves. Let $\tilde{M}$ be the universal covering of M. Denote by Γ the group of deck transformations of $\tilde{M}$. Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\tilde{M}$ and denote by $\mathcal{F}_1$ the closure of $\tilde{\mathcal{F}}$. Since $\tilde{F}$ is invariant under the action of Γ, so is its closure $\mathcal{F}_1$. Thus $\mathcal{F}_1$ induces a singular Riemannian foliation $\mathcal{F}_2$ on M whose leaves contain the leaves of $\mathcal{F}$. Since the leaves of $\mathcal{F}$ are dense so must be the leaves of $\mathcal{F}_2$. In particular, $\mathcal{F}_2$ and, therefore, $\mathcal{F}_1$ must be regular Riemannian foliations. Consider the Riemannian orbifold $B = \tilde{M}/\mathcal{F}_1$. Since $\mathcal{F}_2$ has dense leaves, the natural isometric action of Γ on B must have dense orbits. In particular, B must be a homogeneous Riemannian manifold.

Thus, the projection $p : \tilde{M} \to B$ is a Riemannian submersion. From the long exact sequence of the fibration p (and the contractibility of $\tilde{M}$) we deduce that B must be simply connected. Since B is homogeneous, its homotopy and homology groups are finitely generated. From the long exact sequence of p we deduce that the homotopy groups of the fibers L of p (these are leaves of $\mathcal{F}_1$) are abelian and finitely generated. Hence, the homology groups of L are finitely generated as well. Now, we can apply the spectral sequence for the fiber bundle p, as in[4] p. 599, and deduce that the homology groups of L and B must vanish in positive degrees. We conclude that L and B are contractible.

It remains to prove that the leaves of $\tilde{F}$ are closed, i.e., that $\tilde{\mathcal{F}}$ and $\mathcal{F}_1$ coincide. Assume the contrary and take a non-closed leaf L. Then its closure $\bar{L}$ is a leaf of $\mathcal{F}_1$, hence it is contractible. Thus the restriction of $\tilde{\mathcal{F}}$ to $\bar{L}$ is a Riemannian foliation with dense leaves on a complete, contractible manifold $\bar{L}$. But this is impossible ([24]). This finishes the proof of Proposition 1.1.

Now we can finish the proof of Table 1.2. We already now, that the closure $\bar{\mathcal{F}}$ is a regular Riemannian foliation on M. Moreover, the leaves of $\bar{\mathcal{F}}$ have no focal points, and $M/\bar{\mathcal{F}}$ is a Riemannian orbifold without conjugate points. Now, the proof of[25] Theorem 2 reveals that the lift $\mathcal{F}_1$ of $\bar{\mathcal{F}}$ to the universal covering $\tilde{M}$ is a simple foliation. Moreover, the quotient $\hat{B} = \tilde{M}/\mathcal{F}_1$ is a Riemannian manifold without conjugate points. From the long exact sequence we deduce that $\hat{B}$ is simply connected. Therefore, it is diffeomorphic to $\mathbb{R}^n$. Each leaf L of $\mathcal{F}_1$ has no focal points. Therefore, its normal exponential map is a diffeomorphism. Thus the distance function $d_x : L \to \mathbb{R}$ to each point $x \in M \setminus L$ is a Morse function on L with only one critical point. Therefore, L is diffeomorphic to a Euclidean space as well.

In particular, the leaves of $\bar{\mathcal{F}}$ are aspherical. From Proposition 1.1 we deduce that the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to M has closed and contractible leaves. In particular, all leaves of $\tilde{\mathcal{F}}$ have trivial fundamental group and therefore no holonomy. Therefore, $\tilde{\mathcal{F}}$ is a simple foliation. From the long exact sequence we deduce that the quotient $B_1 = \tilde{M}/\tilde{\mathcal{F}}$ is a contractible manifold.

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