Geometric resolution of singular Riemannian foliations

Alexander Lytchak

Abstract

We prove that an isometric action of a Lie group on a Riemannian manifold admits a resolution preserving the transverse geometry if and only if the action is infinitesimally polar. We provide applications concerning topological simplicity of several classes of isometric actions including polar and variationally complete ones. All results are proven in the general framework of singular Riemannian foliations.

1. Introduction

For an isometric action of a Lie group G on a Riemannian manifold M the presence of singular orbits is the main source of difficulties to understand the geometric and topological properties of the action. It seems natural to look for some procedure resolving the singularities, i.e., some way to pass from M to some other G-manifold $\hat{M}$ with only regular orbits, related to M in some canonical way. For the choice of the procedure it is crucial what kind of information one would like to preserve by this resolution. If one only would like to leave the regular part of the action unchanged, not caring about the singular part, then there is a canonical procedure resolving an arbitrary action. One starts with the most singular stratum, replaces it by the projectivized normal bundle and proceeds inductively. For this topological approach we refer the reader, for instance, to [Was97] or to [Mol84]. The disadvantage of this method is that many crucial geometric and topological properties of the action are "concentrated" in the singular locus and cannot be traced by this procedure.

In geometry, it seems natural to consider the quotient M/G with the induced metric as the essence of the action. Thinking of the action as of a (singular) foliation, one considers the transverse geometry as the most important object. Therefore it seems natural to look only for such *geometric* resolutions $\hat{M}$ with a G-equivariant surjective map $f : \hat{M} \to M$ such that the induced map $f : \hat{M}/G \to M/G$ is an isometry (some partial resolutions of these type have already been considered, for instance in [GS00]). The main technical result of this paper (Theorem 1.1) states that a geometric resolution exists if and only if all isotropy representations of the action are polar. Many natural classes of actions, for instance polar ones, variationally complete ones or arbitrary actions of cohomogeneity at most two satisfy this property of being infinitesimally polar. Moreover, if the action is infinitesimally polar, there is a canonical resolution that inherits many properties of the original action. This provides a way to reduce the study of some topological and geometric properties of actions to the case of regular actions, where they can be easily established; see the subsequent results in the introduction.

It turns out that the action itself does not play a role in our considerations, but only the decomposition of the manifold into orbits, i.e., a *singular Riemannian foliation.* We refer the reader to [Mol88b] or to the preliminaries in Section 2 for basics about singular Riemannian foliations. We mention that geometric desingularisation of (some) polar singular Riemannian foliation was a

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main tool in [Bou95], our work can be seen as a continuation of [Bou95]. Readers only interested in the special case of group actions may just consider all singular Riemannian foliations as orbit decompositions of an isometric group action. We also would like to mention [Wie08], where the ideas of the present paper are elaborated in much more details and simplified in some places in the case of isometric group actions.

Definition 1.1 *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. A geometric resolution of $(M,\mathcal{F})$ is a smooth surjective map $F:\hat M\to M$ from a smooth Riemannian manifold $\hat M$ with a regular Riemannian foliation $\hat{\mathcal{F}}$ such that the following holds true. For all smooth curves γ in $\hat M$ the transverse lengths of γ with respect to $\hat{\mathcal{F}}$ and of $F(\gamma)$ with respect to $\mathcal{F}$ coincide.*

Here the transverse length is defined as usual in the theory of foliations as the length of the projection to local quotients (Subsection 2.6). The last requirement in the definition above means that F sends leaves of $\hat{\mathcal{F}}$ to leaves of $\mathcal{F}$ and induces a length-preserving map between the quotients $F:\hat M/\hat{\mathcal{F}}\to M/\mathcal{F}$, see Section 3. Considering the quotient space $M/\mathcal{F}$ with its local metric structure as the essence of the singular Riemannian foliation $(M,\mathcal{F})$, the above definition becomes the most natural one.

Our main result reads as follows:

Theorem 1.1 *Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Then $(M,\mathcal{F})$ has a geometric resolution if and only if $\mathcal{F}$ is infinitesimally polar. If $\mathcal{F}$ is infinitesimally polar then there is a canonical resolution $F:\hat M\to M$ with the following properties. The resolution $\hat M$ is of the same dimension as M and the map F induces a bijection between the spaces of leaves. Moreover, F is a diffeomorphism, when restricted to the preimage of the set of regular points of $(M,\mathcal{F})$. The map F is proper and 1-Lipschitz. In particular, the resolution $\hat M$ is compact or complete if M has the corresponding property. The isometry group Γ of $(M,\mathcal{F})$ acts by isometries on $(\hat M,\hat{\mathcal{F}})$ and the map $F:\hat M\to M$ is Γ-equivariant. If $\mathcal{F}$ is given by the orbits of a group G of isometries of M then G acts by isometries on $\hat M$, and $\hat{\mathcal{F}}$ is given by the orbits of G. If M is complete then the singular Riemannian foliation $\mathcal{F}$ has no horizontal conjugate points if and only if $\hat{\mathcal{F}}$ has no horizontal conjugate points. If M is complete then the singular Riemannian foliation $\mathcal{F}$ is polar if and only if $\hat{\mathcal{F}}$ is polar.*

The infinitesimal polarity of $\mathcal{F}$ means that locally the singular Riemannian foliation $\mathcal{F}$ is equivalent to an isoparametric singular Riemannian foliation on a Euclidean space (Subsection 2.5). For polar singular Riemannian foliations we refer to Subsection 2.4 (cf. [Ter85], [Bou95], [Ale04], [Ale06]) and for singular Riemannian foliations without horizontal conjugate points we refer to [BS58], [LT07b] [LT07a].

Before we are going to comment on this theorem and related results, we state some consequences that motivated our study of geometric resolutions. Recall that a (regular) Riemannian foliation $\mathcal{F}$ on a Riemannian manifold M is called simple if it is given by the fibers of a Riemannian submersion. If M is complete (or, more generally, if $\mathcal{F}$ is full, see Section 5) then $\mathcal{F}$ is simple if and only if all leaves of $\mathcal{F}$ are closed and have no holonomy ([Her60]). The next result generalizes [BH83] and [Heb86], Theorem 2 to the realm of singular Riemannian foliations.

Theorem 1.2 *Let M be a complete, simply connected Riemannian manifold, and let $\mathcal{F}$ be a singular Riemannian foliation on M. If $\mathcal{F}$ is polar, or if $\mathcal{F}$ has no horizontal conjugate points then the leaves of $\mathcal{F}$ are closed. Moreover, the restriction of $\mathcal{F}$ to the regular part of M is a simple foliation.*

Remark 1.1 *In [Ter85] it is shown that isoparametric foliations on simply connected spaces of constant curvature have closed leaves and that there are no exceptional leaves, i.e., that all regular leaves* *have trivial holonomy. In [TÖ6] it is shown that if $\mathcal{F}$ is a polar singular Riemannian foliation on a simply connected symmetric space M then properness of all leaves implies vanishing of holonomy of regular leaves. Finally, in [AT06] the same result was shown for an arbitrary complete, simply connected space M. Thus, in the case of polar singular Riemannian foliations only the closedness of $\mathcal{F}$ is new. See also Theorem 1.6 and Remark 1.3 below for more general statements.*

Since a connected group of isometries of a Riemannian manifold is closed if and only its orbits are closed, Theorem 1.2 reads in the case of group actions as follows:

Corollary 1.3 *Let M be a complete, simply connected manifold and let a connected group G act by isometries of M. If the action is polar or variationally complete then the image of G in the isometry group of M is closed and there are no exceptional orbits of the action.*

From Theorem 1.2 and [LT07a], Theorem 1.7 we immediately get a complete description of singular Riemannian foliations without horizontal conjugate points in terms of their quotient spaces. Since singular Riemannian foliations without horizontal conjugate points generalize the concept of variationally complete actions introduced in [Bot56] and [BS58] and investigated in [Con72],[GT02], [DO01] and [LT07b], the next result also gives a description of variationally complete actions in terms of the quotient spaces. Since complete non-negatively curved Riemannian orbifolds without conjugate points are flat, the next result generalizes the main results of [DO01], [GT02] and [LT07b].

Corollary 1.4 *Let M be a complete Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation. Then $\mathcal{F}$ does not have horizontal conjugate points if and only if the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal covering $\tilde{M}$ of M is closed and the quotient $\tilde{M}/\tilde{\mathcal{F}}$ is a Riemannian orbifold without conjugate points.*

To deduce Theorem 1.2 from Theorem 1.1 we proceed as follows. If $\mathcal{F}$ is polar then $\mathcal{F}$ is also infinitesimally polar. If $\mathcal{F}$ has no horizontal conjugate points then it is infinitesimally polar as well, due to [LT07a], Theorem 1.7. Thus we may apply Theorem 1.1 and obtain a regular Riemannian foliation $\hat{\mathcal{F}}$ on a complete Riemannian manifold $\hat{M}$ that is polar or has no horizontal conjugate points. In the first case we apply [BH83] and deduce that the lift of $\hat{\mathcal{F}}$ to the universal covering of $\hat{M}$ is a simple Riemannian foliation. In the second case, the leaves of the regular Riemannian foliation $\hat{\mathcal{F}}$ on the complete Riemannian manifold $\hat{M}$ have no focal points and the proof of [Heb86], Theorem 2 shows that the lift of $\hat{\mathcal{F}}$ to the universal covering of $\hat{M}$ is again a simple Riemannian foliation. But $(\hat{M}, \hat{\mathcal{F}})$ coincides with $(M, \mathcal{F})$ on the regular part M_0 of M. Therefore, the restriction of $\mathcal{F}$ to M_0 becomes simple, when lifted to the universal covering $\tilde{M}_0$ of M_0. Thus, Theorem 1.2 follows from the next general topological observation whose proof will be given in Section 5. The proof of this result is implicitly contained in [Mol88b], p.213-214 (see also [Mol88a]).

Theorem 1.5 *Let M be a complete, simply connected Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. If the restriction of $\mathcal{F}$ to the regular part M_0 becomes simple, when lifted to the universal covering $\tilde{M}_0$ of M_0, then the restriction of $\mathcal{F}$ to M_0 is a simple foliation.*

The combination of the last theorem and Theorem 1.1 allows us to describe singular Riemannian foliations on simply connected manifolds that have *good* Riemannian orbifolds as quotients. In the formulation we will use the notion of *Coxeter orbifolds* (cf. [DAM07]. These are Riemannian orbifolds locally diffeomorphic to Weyl chambers of Euclidean Coxeter groups.

Theorem 1.6 *Let M be a complete, simply connected Riemannian manifold and let $\mathcal{F}$ be a closed singular Riemannian foliation on M with quotient $B = M/\mathcal{F}$. Then the following are equivalent.*

i) *B is a good Riemannian orbifold;*

ii) *B is a Riemannian Coxeter orbifold;*

iii) *$\mathcal{F}$ is infinitesimally polar and all regular leaves of $\mathcal{F}$ have trivial holonomy.*

Example 1.2 *Closed singular Riemannian foliations that are polar or have no horizontal conjugate points have good Riemannian orbifolds as quotients. In the case of closed polar singular Riemannian foliations on simply connected manifolds it was shown in [AT06], that the quotients are Coxeter orbifolds.*

Remark 1.3 *The proof of Theorem 1.6 reveals the following slightly more general statement. Namely, let M be complete and simply connected and let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation on M. Let $(\hat{M}, \hat{\mathcal{F}})$ be the resolution of $(M, \mathcal{F})$. If the groupoid of local isometries associated with the foliation $\hat{\mathcal{F}}$ is developable (cf. [Hae88]) then $\mathcal{F}$ is closed, regular leaves of $\mathcal{F}$ have trivial holonomy and $M/\mathcal{F}$ is a Coxeter orbifold.*

In view of Theorem 1.6 it seems natural to ask the following

Question 1.4 *What simply connected Coxeter orbifolds B can be represented as quotient spaces $B = M/\mathcal{F}$ for some singular Riemannian foliation $\mathcal{F}$ on some simply connected Riemannian manifold M.*

Combining Theorem 1.6 and [LT07a], Theorem 1.1, we get a characterization of actions on Riemannian manifolds with good orbifolds as quotients. Using this characterization it seems possible to find all isometric actions on Euclidean spheres with good orbifolds as quotients. We hope to address this issue in a forthcoming paper.

Corollary 1.7 *Let M be a complete, simply connected Riemannian manifold. Let G be a closed, connected group of isometries of M. Let B be the quotient $B = M/G$. For $x \in M$, denote by K_x the isotropy group at x, by K_x^0 its connected component, and by H_x the normal space to the orbit Gx at x. The quotient B is a Riemannian orbifold if and only if all isotropy representations of K_x on H_x are polar. In such a case the following are equivalent:*

i) *B is a good Riemannian orbifold;*

ii) *The orbits of K_x and of K_x^0 on H_x coincide;*

iii) *There are no exceptional orbits of the action.*

The proof of Theorem 1.1 is provided in Section 4 and Section 3 along the following lines. For an infinitesimally polar $\mathcal{F}$ on a Riemannian manifold M one uses the ideas of [Bou95] and [TÖ6] and defines the resolution $\hat{M}$ to be the subset of the Grassmannian bundle $Gr_k(M)$ consisting of all infinitesimal horizontal sections of $\mathcal{F}$. In the polar case the result is contained in [Bou95] and [TÖ6]. In the general case one follows an idea from [LT07a] and uses transformation relating horizontal geometry of different Riemannian metrics adapted to a given foliation to reduce the question to the polar case. Unfortunately, one has to be quite careful about some essential technical difficulties concerning the smoothness of objects defined in the course of the proof.

Remark 1.5 *The proof shows (and is based on) the fact that the resolution $(\hat{M}, \hat{\mathcal{F}})$ considered as a foliation on a manifold (disregarding the Riemannian metric on $\hat{M}$) does not depend on the Riemannian metric adapted to the singular Riemannian foliation $\mathcal{F}$ on M.*

To see that a singular Riemannian foliation $\mathcal{F}$ with a metric resolution $\hat{\mathcal{F}}$ is infinitesimally polar one observes that in a regular Riemannian foliation transversal sectional curvatures remain bounded on compact subsets. Now, one uses the transverse equivalence of $\mathcal{F}$ and $\hat{\mathcal{F}}$ and deduces from [LT07a], Theorem 1.4 that this property characterize infinitesimally polar singular Riemannian foliations. This already proves the claim in the case of a compact resolution $\hat{M}$. In the general case one needs to be more careful and to prove some extensions of results in [LT07a], that may also be of independent interest (Lemma 3.4 and Proposition 3.5)

We would like to mention that Sections 3, 4 and 5 do not depend on each other. Thus, reader only interested in Theorem 1.5 may directly proceed to Section 5 and reader only interested in the (more important) if part of Theorem 1.1 may skip Section 3.

2. Preliminaries

2.1 Singular Riemannian foliations

A *transnormal system* $\mathcal{F}$ on a Riemannian manifold M is a decomposition of M into smooth injectively immersed connected submanifolds, called leaves, such that geodesics emanating perpendicularly to one leaf stay perpendicularly to all leaves. A transnormal system $\mathcal{F}$ is called a *singular Riemannian foliation* if there are smooth vector fields X_i on M such that for each point $p \in M$ the tangent space $T_pL(p)$ of the leaf $L(p)$ through p is given as the span of the vectors $X_i(p) \in T_pM$. We refer to [Mol88b] and [Wil07] for more on singular Riemannian foliations. Examples of singular Riemannian foliations are (regular) Riemannian foliations and the orbit decomposition of an isometric group action.

2.2 Stratification

Let $\mathcal{F}$ be a singular Riemannian foliation on the Riemannian manifold M. The *dimension of* $\mathcal{F}$, $\dim(\mathcal{F})$, is the maximal dimension its leaves. The *codimension of* $\mathcal{F}$, $\mathrm{codim}(\mathcal{F}, M)$, is defined by $\dim(M) - \dim(\mathcal{F})$. For $s \leq \dim(\mathcal{F})$, denote by Σ_s the subset of all points $x \in M$ with $\dim(L(x)) = s$. Then Σ_s is an embedded submanifold of M and the restriction of $\mathcal{F}$ to Σ_s is a Riemannian foliation. For a point $x \in M$, we denote by Σ^x the connected component of Σ_s through x, where $s = \dim(L(x))$. We call the decomposition of M into the manifolds Σ^x the *canonical stratification* of M.

The subset $\Sigma_{\dim(\mathcal{F})}$ is open, dense and connected in M. It is the *regular stratum* M. It will be denoted by M_0 and will also be called the set or regular points of M. All other strata Σ^x, called *singular strata*, have codimension at least 2 in M. For any singular stratum Σ, we have $\mathrm{codim}(\mathcal{F}, \Sigma) < \mathrm{codim}(\mathcal{F}, M)$.

2.3 Infinitesimal singular Riemannian foliations

Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Let $x \in M$ be a point. Then there is a well defined singular Riemannian foliation $T_x\mathcal{F}$ on the Euclidean space (T_xM, g_x) with the following properties:

i) There is a neighborhood O of x and a diffeomorphic embedding $\phi : O \to T_xM$, with $D_x\phi = Id$ and $\phi^*(T_x\mathcal{F}) = \mathcal{F}|_O$.

ii) $T_x\mathcal{F}$ is homogeneous, i.e., for each non-zero real number λ, the multiplication by λ on T_xM preserves $T_x\mathcal{F}$.

iii) The singular foliation $T_x\mathcal{F}$ on the tangent space T_xM does not depend on the Riemannian metric adapted to $\mathcal{F}$.

The singular Riemannian foliation $T_x\mathcal{F}$ on the tangent space T_xM will be called the *infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point x.*

2.4 Horizontal sections

We refer to [Bou95], [Ale04], [Ale06] for more on polar singular Riemannian foliations. Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. A global (local) horizontal section through x is a smooth immersed submanifold $x \in N \subset M$ that intersects all leaves of $\mathcal{F}$ (all leaves in a neighborhood of x), such that all intersections are orthogonal. $\mathcal{F}$ is called polar (locally polar) if there are (local) global horizontal sections through every point $x \in M$. Each local section N of a singular Riemannian foliation is totally geodesic. Moreover, for each $x \in N$, $T_xN \subset T_xM$ is a horizontal section of the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$. On the other hand, if $\mathcal{F}$ is locally polar then each horizontal section $V \subset T_xM$ of the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$ is the tangent space to a local horizontal section of $\mathcal{F}$.

Recall, that a singular Riemannian foliation $\mathcal{F}$ is locally polar if and only if the restriction of $\mathcal{F}$ to the regular part M_0 has integrable horizontal distribution ([Ale06]). Moreover, a locally polar singular Riemannian foliation on a complete Riemannian manifold is polar.

2.5 Infinitesimal polarity

The singular Riemannian foliation $\mathcal{F}$ is called infinitesimally polar at the point $x \in M$ if the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$ is polar. We say that $\mathcal{F}$ is infinitesimally polar if it is infinitesimally polar at all points. In [LT07a] it is shown that $\mathcal{F}$ is infinitesimally polar at the point x if and only if for all sequences x_i of regular points converging to x, the supremum $\bar{\kappa}(x_i)$ of the sectional curvatures at projections of x_i to local quotients remain bounded away from infinity. Another equivalent condition derived in [LT07a], is that $\mathcal{F}$ is locally closed at x and that local quotients at x are smooth Riemannian orbifolds.

2.6 Transverse length

Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. For x in M, we denote by V_x the tangent space to the leaf $V_x = T_xL(x)$ and call it the *vertical space at* x. The orthogonal complement of V_x will be denoted by H_x (or by $H_x(g)$, if we want to specify the Riemannian metric g). This subspace H_x will be called the *horizontal subspace at* x. By $P_x : T_x \to H_x$ we denote the orthogonal projection. The spaces H_x vary semi-continuously. Therefore, for each smooth curve γ in M, the value $L_{hor}(\gamma) := \int |P_{\gamma(t)}(\gamma'(t))| dt$ is well defined. We call this quantity the *transversal length of* γ. If $B = M/\mathcal{F}$ is a Hausdorff metric space then $L_{hor}(\gamma)$ is the length of the projection of γ to B. Note that a smooth curve has transversal length zero if and only it is completely contained in one leaf.

3. The only if part

We are going to prove the only if part of the first statement of Theorem 1.1 in this section. Thus, let $\hat{\mathcal{F}}$ be a regular Riemannian foliation on a Riemannian manifold $\hat{M}$, let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M and let $F : \hat{M} \to M$ be a geometric resolution. We are going to analyze F and to prove that $\mathcal{F}$ is infinitesimally polar. The proof in the case of compact $\hat{M}$ was explained in the introduction. In the general case, we will give a proof along the same lines, but the proof becomes technically more involved.

First of all, F sends curves of zero transversal length to curves of zero transversal length, therefore F sends leaves into leaves, i.e., $F(\hat{L}(x)) \subset L(F(x))$ for all $x \in \hat{M}$.

For each open subset O of M the restriction $F : F^{-1}(O) \to O$ is again a geometric resolution. As usual, let M_0 denote the set of regular points of M and set $\tilde{M} := F^{-1}(M_0)$. Since the restriction of $\mathcal{F}$ to M_0 is a *regular* Riemannian foliation, we deduce from continuity reasons, that for all $x \in \tilde{M}$

the map $G_x := P_{F(x)} \circ D_x F : H_x \to H_{F(x)}$ is an isometric embedding. Here, the horizontal subspaces H and the projections P are defined as in Subsection 2.6.

On the other hand, F is smooth and surjective. By Sards theorem there is at least one point $x \in \tilde{M}$ such that $D_x F : T_x \tilde{M} \to T_{F(x)} M$ is surjective. Since $D_x F$ sends $T_x(L(x))$ to a subspace of $T_{F(x)}(L(F(x)))$ we deduce that the map $G_x : H_x \to H_{F(x)}$ must be surjective at such points. Therefore, $\dim(H_x) = \dim(H_{F(x)})$. Hence, $\operatorname{codim}(M, \mathcal{F}) = \operatorname{codim}(\hat{M}, \hat{\mathcal{F}})$. Moreover, for each $x \in \tilde{M}$, the map $G_x : H_x \to H_{F(x)}$ is an isometry.

Thus, for each point $x \in \tilde{M}$, we find a small neighborhood O of x such that $\hat{\mathcal{F}}$ on O is given by a Riemannian submersion $s_1 : O \to B_1$, such that $\mathcal{F}$ on $F(O)$ is given by a Riemannian submersion $s_2 : F(O) \to B_2$, and such that F induces an isometry $\bar{F} : B_1 \to B_2$ between the local quotients.

This finishes the analysis of F on $\tilde{M}$. The picture over the singular points is much more complicated; for instance, F is usually not open at points outside of $\tilde{M}$. We start our discussion of the singular part with the following easy observation.

Lemma 3.1 *Let $\gamma_1 : [0,a] \to \hat{M}$ and $\gamma_2 : [0,a] \to M$ be horizontal geodesics with $\gamma_2((0,a]) \subset M_0$. If $F(\gamma_1(t)) \subset L(\gamma_2(t))$, for all t, then the sectional curvatures in local quotients at $L(\gamma_2(t))$, $t \in (0,a]$, are uniformly bounded.*

Proof. From the discussion above we know that the sectional curvatures in local quotients at $L(\gamma_1(t))$ and $\hat{L}(\gamma_2(t))$ coincide for all $t \in (0,a]$. Since $[0,a]$ is compact and $\hat{F}$ is a *regular* Riemannian foliation, the sectional curvatures in local quotients at $\hat{L}(\gamma_2(t))$ are uniformly bounded. □

The idea is now to find such curves starting at all points and to deduce infinitesimal polarity from this existence.

Lemma 3.2 *The open subset $\tilde{M}$ is dense in $\hat{M}$.*

Proof. Assume the contrary and choose an open subset O of $\hat{M} \setminus \tilde{M}$. By making O smaller we may assume that $F(O)$ is contained in a singular stratum Σ of M. Now, the restriction of $\mathcal{F}$ to Σ is again a *regular* Riemannian foliation. Thus, for each $x \in O$, we obtain by continuity that $D_x F$ maps H_x injectively onto the subspace $D_x F(H_x)$ that intersects $T_{F(x)}(L(F(x))$ only in $\{0\}$. Thus we deduce

$$\operatorname{codim}(\hat{M}, \hat{\mathcal{F}}) = \dim(H_x) \leq \operatorname{codim}(\Sigma, \mathcal{F}) < \operatorname{codim}(M, \mathcal{F})$$

since Σ is a singular stratum. This contradicts the previously obtained equality $\operatorname{codim}(\hat{M}, \hat{\mathcal{F}}) = \operatorname{codim}(M, \mathcal{F})$. □

Now we can prove:

Lemma 3.3 *For each $x \in M$, there are horizontal geodesics $\gamma_1 : [0,a] \to \hat{M}$ and $\gamma_2 : [0,a] \to M$ such that $\gamma_2(0) = x$, $\gamma_2((0,a] \subset M_0$ and $F(\gamma_1(t)) \subset L(\gamma_2(t))$, for all t.*

Proof. Choose a distinguished tubular neighborhood U at x and a preimage y of x in $\hat{M}$. Make the diameter ϵ of U so small that all geodesics starting in the ϵ-neighborhood O of y are defined at least for the time ϵ. Take a point $z \in \tilde{M} \cap O$ with $\bar{z} = F(z) \in U$. Let $\bar{x}$ be the projection of $\bar{z}$ onto the leaf of $\mathcal{F}$ through x in U. Then $\bar{x}$ is the only possibly non-regular point on the geodesic $\gamma_3 = \bar{z}\bar{x}$. Consider the horizontal geodesic γ_1 in $\hat{M}$ starting at z in the direction h with $G_z(h) = \gamma_3'$. From the understanding of F on $\tilde{M}$, we deduce that $F(\gamma_1(t))$ is contained in $L(\gamma_3(t))$ for all $t \in [0, d(\bar{z}, \bar{x})]$. Now, replacing γ_3 through a horizontal geodesic starting in a point on $L(\bar{z})$ and ending in x, we obtain a horizontal geodesic γ_2 ending in x with $F(\gamma_1(t)) \in L(\gamma_2(t))$. It remains to reverse the orientations of γ_1 and γ_2. □

Now the proof of the infinitesimal polarity of $\mathcal{F}$ is finished by combining Lemma 3.1, Lemma 3.3 and the following lemma, that we consider to be of independent interest.

Lemma 3.4 *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. Let $x \in M$ be a point. Let $\gamma : [0, \epsilon] \to M$ be a horizontal geodesic starting at x, such that $\gamma((0, \epsilon])$ is contained in the set of regular points M_0. If all sectional curvatures in local quotients are uniformly bounded along $\gamma(0, \epsilon]$ then $\mathcal{F}$ is infinitesimally polar at x.*

Proof. Consider $T_x\mathcal{F}$ as the limit of rescaled singular Riemannian foliations $(M, \mathcal{F})$ as in [LT07a], p.10. As in [LT07a], we deduce that $T_x\mathcal{F}$ is a singular Riemannian foliation on the Euclidean space T_xM such that at the regular point $v = \gamma'(0) \in T_xM$ all sectional curvatures vanish in local quotients. In this case, Proposition 3.5 below implies that $T_x\mathcal{F}$ is polar. □

Proposition 3.5 *Let $\mathcal{F}$ be a singular Riemannian foliation on the Euclidean space $\mathbf{R}^n$. Let L be a regular leaf such that in local quotients all sectional curvatures vanish at the image of this leaf. Then $\mathcal{F}$ is polar.*

Proof. Since $\mathbf{R}^n$ is flat, the sectional curvatures at the point $\{L\}$ in local projections vanish if and only if the O'Neill tensor $A : H_x \times H_x \to T_x(L(x))$ vanishes identically at all points $x \in L$. But this implies that each Bott-parallel normal field H along L is a parallel normal field. Since all these fields are equifocal (cf. [AT08]), we get that L is an isoparametric submanifold of $\mathbf{R}^n$ and that $\mathcal{F}$ coincides with the isoparametric foliation defined by the isoparametric submanifold L. □

4. Desingularization

4.1 Notations

First, let T be a finite-dimensional real vector space with scalar products g and g^+. Let $A : T \to T$ be the linear map defined by $g^+(A(v), w) = g(v, w)$ for all $v, w \in T$. Then, for each linear subspace H of T, the image $A(H)$ of H satisfies $H^{\perp_g} = (A(H))^{\perp_{g^+}}$, i.e., the g-orthogonal complement of H coincides with the g^+-orthogonal complement of $A(H)$. We will denote the map A by I_{g,g^+}. By the same symbol I_{g,g^+} we denote the induced map on the Grassmannians $Gr_k(T)$, i.e., on the spaces of k-dimensional linear subspaces of T. Note that $I_{g,g^+} \circ I_{g^+,g} = \mathrm{Id}$.

If M is a Riemannian manifold with Riemannian metrics g, g^+ then we get a bundle automorphism $I_{g,g^+} : TM \to TM$ of the tangent bundle TM of M. For $k \geq 0$, we denote by $Gr_k = Gr_k(M)$ the Grassmannian bundle of the tangent bundle of M, i.e., the bundle of k-dimensional subspaces of tangent spaces of M. By the same symbol I_{g,g^+} we will denote the induced bundle automorphism $I_{g,g^+} : Gr_k \to Gr_k$.

Let now $\mathcal{F}$ be a singular foliation adapted to the Riemannian metrics g and g^+, i.e., $\mathcal{F}$ is a singular Riemannian foliation with respect to the Riemannian metrics g and g^+. For any point $x \in M$, we have the subspaces $H_x(g)$ and $H_x(g^+)$ of g-horizontal and of g^+-horizontal vectors, respectively. By construction, our transformation I_{g,g^+} satisfies $I_{g,g^+}(H_x(g)) = H_x(g^+)$, since H_x is defined as orthogonal complement of the vertical space V_x that does not depend on the adapted Riemannian metric.

4.2 Basic construction

Let (M, g) be a Riemannian manifold and let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation on M of codimension k.

We denote by $\hat{M} \subset Gr_k$ the set of all k-dimensional infinitesimal sections of $\mathcal{F}$. Thus $p^{-1}(x) \subset \hat{M}$ is the manifold of horizontal sections of the polar Riemannian foliation $T_x\mathcal{F}$ on T_xM. In particular, for each regular point $x \in M_0 \subset M$, the preimage $p^{-1}(x)$ consists of only one point $H_x \in Gr_kM$.

We are going to prove:

i) $\hat{M}$ is a closed smooth submanifold of Gr_k.

ii) The decomposition of $\hat{M}$ into preimages $\hat{L} = p^{-1}(L)$ of the leaves of $\mathcal{F}$ is a smooth foliation $\hat{\mathcal{F}}$ of $\hat{M}$.

The definition of $\hat{M}$ and of $\hat{\mathcal{F}}$ are local on M and so are the claims. Thus we may restrict ourselves to a small distinguished neighborhood U of a given point $x \in M$. Pulling back the flat metric on T_xM by the diffeomorphism ϕ, we thus reduce the question to the following situation, to which we will refer later as the *standard case.* The manifold M is an open subset of the Euclidean space $\mathbf{R}^n$ with a flat (constant) Riemannian metric g^+, and $\mathcal{F}$ is the restriction of an isoparametric foliation on $\mathbf{R}^n$ to M. Moreover, g is a Riemannian metric on M adapted to $\mathcal{F}$.

Let $\hat{M}^+$ be the subset of the Grassmannian Gr_k of all infinitesimal horizontal sections of $\mathcal{F}$ with respect to the Riemannian metric g^+. Moreover, by $\hat{\mathcal{F}}^+$ we denote the decomposition of $\hat{M}^+$ into preimages of leaves of $\mathcal{F}$. Due to [Bou95], $\hat{M}^+$ is a closed submanifold of Gr_k and $\hat{\mathcal{F}}^+$ is a foliation on $\hat{M}^+$. (In fact, we only use the result of Boualem in the case of an isoparametric foliation on the flat $\mathbf{R}^n$).

We claim that the gauge $I_{g,g^+} : Gr_k \to Gr_k$ sends M to $\hat{M}^+$. As soon as the claim is verified, we deduce that I_{g,g^+} sends $\hat{\mathcal{F}}$ to $\hat{\mathcal{F}}^+$, because I_{g,g^+} is a bundle morphism, i.e., it commutes with the projection p. Thus this claim would imply that $\hat{M}$ is a smooth closed submanifold and that $\hat{\mathcal{F}}$ is a foliation on $\hat{M}$.

Thus it remains to prove the following

Lemma 4.1 *Let M be a manifold and let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation with respect to Riemannian metrics g and g^+. Then $I_{g,g^+} : Gr_k \to Gr_k$ sends $\hat{M}$ to $\hat{M}^+$.*

Proof. Choose a point $x \in M$. The singular foliation $T_x\mathcal{F}$ on the tangent space T_xM is defined independently of g and g^+. The preimages of x in $\hat{M}$ and in $\hat{M}^+$ are defined only in terms of $T_x\mathcal{F}$, g_x and g_x^+, thus it is enough to prove the claim for the case $M = \mathbf{R}^n$, where $\mathcal{F}$ is a polar singular Riemannian foliation with respect to the flat metrics g and g^+ (by replacing $\mathcal{F}$ through $T_x\mathcal{F}$). In this case $\hat{M}$ and $\hat{M}^+$ are closed submanifolds of Gr_kM and the regular part $p^{-1}(M_0)$ is open and dense in both $\hat{M}$ and $\hat{M}^+$ ([Bou95]). By definition, I_{g,g^+} sends $p^{-1}(M_0) \cap \hat{M}$ to $p^{-1}(M_0) \cap \hat{M}^+$.

By continuity, we deduce $I_{g,g^+}(\hat{M}) \subset \hat{M}^+$. Reversing the role of g and g^+ and using that $I_{g,g^+} \circ I_{g^+,g} = Id$, we deduce $I_{g,g^+}(\hat{M}) = \hat{M}^+$. □

4.3 Regular vectors

Before we are going to define a Riemannian structure on $\hat{M}$, we will need some observations concerning the space of horizontal vectors. Let $\mathcal{F}$ be again a singular Riemannian foliation on a Riemannian manifold (M, g). As in [LT07a], we denote by $D(g)$ the space of all unit horizontal vectors on M. By $D^0 = D^0(g) \subset D(g)$ we denote the space of all *regular horizontal vectors.* Recall, that a horizontal vector $v \in H_x$ is called regular if the horizontal geodesic γ^v starting in the direction of v contains at least one regular point ([LT07a]). In this case all but discretely many points on γ^v are regular. Equivalently, one can say that a vector $v \in H_x$ is regular, if $v \in T_xM$ is a regular point of the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$. Recall finally, that D^0 is a smooth, injectively immersed submanifold of the unit tangent bundle U^gM of M, that is invariant under the geodesic flow.

If $\mathcal{F}$ is infinitesimally polar then a horizontal vector v is regular if and only if it is contained in only one horizontal section S of the isoparametric foliation $T_x\mathcal{F}$. The assignment of the section S

to the regular horizontal vector v defines a map $m = m(g) : D^0 \to \hat{M}$. We are going to prove that m is a smooth submersion.

First, recall that for another Riemannian metric g^+ adapted to $\mathcal{F}$ we have an induced map $I_{g,g^+} : D(g) \to D(g^+)$ that is the restriction of the smooth map $I_{g^+,g}$ between the unit tangent bundles $I_{g,g^+} : U^g M \to U^{g^+} M$ (induced by the fiberwise linear isomorphisms $I_{g,g^+} : TM \to TM$).

Lemma 4.2 *Let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation with respect to the Riemannian metrics g and g^+. Then the map $I_{g,g^+} : D(g) \to D(g^+)$ sends $D^0(g)$ to $D^0(g^+)$.*

Proof. Since I_{g,g^+} sends infinitesimal g-horizontal sections containing a g-horizontal vector v to infinitesimal g^+-horizontal sections containing the g^+-horizontal vector $I_{g,g^+}(v)$, the result follows from the characterization of D^0 as the set of all horizontal vectors, contained in precisely one infinitesimal horizontal section. □

Question 4.1 *Is the statement of the last lemma true for general singular Riemannian foliations, that are not infinitesimally polar?*

Let $M, \mathcal{F}, g, g^+$ be as in the lemma above, and let $\hat{M}$ and $\hat{M}^+$ be the manifolds of horizontal infinitesimal sections with respect to g and g^+ respectively. We have the diffeomorphisms $I_{g,g^+} : D^0(g) \to D^0(g^+)$ and $I_{g^+,g} : \hat{M}^+ \to \hat{M}$ and the maps $m(g) : D^0(g) \to \hat{M}$ and $m(g^+) : D^0(g^+) \to \hat{M}^+$. By construction, the maps commute, i.e., $m(g) = I_{g^+,g} \circ m(g^+) \circ I_{g,g^+}$. Therefore, $m(g)$ is a smooth submersion if and only if $m(g^+)$ is a smooth submersion. Now we can prove:

Lemma 4.3 *Let $\mathcal{F}$ be an infinitesimal Riemannian foliation on a Riemannian manifold (M, g). Then the map $m(g) : D^0(g) \to \hat{M}$ is a smooth submersion.*

Proof. The objects $m(g), D^0, \hat{M}$ are defined locally on M. Thus it is enough to prove the statement in a neighborhood of each point x in M. This reduces the question to the *standard case*. Then the observation preceding this proposition reduces the question to the case $\mathcal{F} = T_x\mathcal{F}$. Thus we may assume that M is the Euclidean space $\mathbf{R}^n$ and that $\mathcal{F}$ is a polar singular Riemannian foliation on $\mathbf{R}^n$.

In this case the claim can be deduced as follows. Given a regular horizontal vector $v \in D^0$, choose a small number ϵ and a neighborhood O of v in D^0 such that $p(\phi_\epsilon(O))$ is contained in the set of regular points of M. Here, $p : UM \to M$ is the projection from the unit tangent bundle to M and ϕ_t is the geodesic flow. The Grassmannian bundle of $\mathbf{R}^n$ is a trivial bundle with a canonical trivialization. With respect to this trivialization we have $m(v) = m(\phi_t(v))$ for all $v \in D^0$ and all t. Thus m is preserved by the geodesic flow ϕ, and the above choice of O reduces the question to the regular part of M. However, in the regular part M_0 of M the claim is clear. □

4.4 Normal distribution

We are going to define now, what is going to be the normal distribution of the foliation $\hat{\mathcal{F}}$ with respect to the Riemannian metric $\hat{g}$ to be defined later. Let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation on a Riemannian manifold M. (Since we are not going to use auxiliary metrics g^+ anymore, we are going to suppress the Riemannian metric g in the sequel). Let $\hat{M}$ be defined as in Subsection 4.2. Let M_0 be the regular part of M and let $\hat{M}_0$ be the preimage $p^{-1}(M_0)$. The restriction $p : \hat{M}_0 \to M_0$ is a diffeomorphism, thus on $\hat{M}_0$ there is a smooth distribution $\hat{\mathcal{H}}_0$ that is sent by p to the horizontal distribution of the Riemannian foliation $\mathcal{F}$ on the Riemannian manifold M_0. We claim:

Lemma 4.4 *There is a unique smooth k-dimensional distribution $\hat{\mathcal{H}}$ on $\hat{M}$ that extends $\hat{\mathcal{H}}_0$.*

Proof. The uniqueness is clear, since $\hat{M}_0$ is dense in $\hat{M}$. In order to prove the existence, it is enough to show that for each element $S \in \hat{M}$ there are k linearly independent smooth vector fields W_i defined on an open neighborhood O of S in $\hat{M}$, such that the restriction of each W_i to $O \cap \hat{M}_0$ is a section of $\hat{\mathcal{H}}_0$.

Thus, let $S \in \hat{M}$ be given and let $x = p(S) \in M$ be the foot point of S. Let $w \in T_xM$ be a regular unit horizontal vector contained in S. Since the map $m : D^0 \to \hat{M}$ is a smooth submersion, we find an open neighborhood O of S in $\hat{M}$ and a smooth section $n : O \to D^0$ with $m \circ n = \mathrm{Id}$ and $n(S) = w$.

Let I be a small interval around 0. Consider the map $\bar{\xi} : O \times I \to D^0$ given by $\bar{\xi}(\bar{S}, t) = \phi_t(n(\bar{S}))$, where ϕ_t denote the restriction of the geodesic flow to D^0. By construction, $\bar{\xi}$ is a smooth map. This implies smoothness of the composition $\xi : O \times I \to O$ given by $\xi = m \circ \bar{\xi}$. By construction, $\xi(\bar{S}, 0) = \bar{S}$ for all $\bar{S} \in O$. Therefore, the map

$$W(\bar{S}) := \frac{d}{dt}\xi(\bar{S}, t)$$

is a smooth vector field on O.

Now, the map $m : D^0 \to \hat{M}$ commutes with the projections to M, i.e., $p(m(v)) = p(v)$ for all $v \in D^0$. Thus the projection of any ξ-trajectory to M is the projection of the corresponding $\bar{\xi}$ trajectory to M. By definition, $\bar{\xi}$-trajectories are flow lines of the geodesic flow. Thus the ξ-trajectory of a point $\bar{S} \in O$ is sent by the projection $p : \hat{M} \to M$ to the regular horizontal geodesic that starts at p in the direction $n(\bar{S})$. In particular, we deduce that the restriction of W to M_0 is a section of $\hat{H}_0$. Moreover, by construction, $p_*(W(\bar{S})) = w$.

Now, we choose a basis w_i of S that consist of regular vectors and applying the above construction, we get the linearly independent smooth vector fields W_i, we were looking for. □

4.5 Riemannian structure

Now we are in position to define the right Riemannian structure $\hat{g}$ on $\hat{M}$. We start with the canonical Riemannian metric h on the Grassmannian bundle $Gr_k(M)$ (cf. [Tö6] for its definition and properties) and denote by the same letter h its restriction to the submanifold $\hat{M}$. The projection $p : (Gr_k, h) \to (M, g)$ is a Riemannian submersion. In particular, the restriction $p : (\hat{M}, h) \to (M, g)$ is 1-Lipschitz.

Let $\hat{\mathcal{H}}$ be the distribution of k-dimensional spaces on $\hat{M}$ defined in the previous subsection. In the proof of Lemma 4.4 we have seen that for each $S \in \hat{M}$ it is possible to choose a base $W_1, ..., W_n$ of $\hat{\mathcal{H}}(S)$ that are mapped by the differential p_* to a base of $S \subset T_{p(S)}M$. In particular, for each $S \in \hat{M}$, the restriction of $p_* : T_S\hat{M} \to T_{p(S)}M$ sends $\hat{\mathcal{H}}(S)$ bijectively to $S \subset T_{p(S)}M$. Since S is normal to the leaf of $\mathcal{F}$ through $p(S)$, we deduce that $\hat{\mathcal{H}}$ and $\hat{\mathcal{F}}$ are transversal.

Now we define the Riemannian metric $\hat{g}$ on $\hat{M}$ uniquely by the following three properties. On $\hat{\mathcal{F}}$ we let $\hat{g}$ coincide with the canonical metric h. We require $\hat{\mathcal{F}}$ and $\hat{\mathcal{H}}$ to be orthogonal with respect to $\hat{g}$. Finally, on $\hat{\mathcal{H}}$ we define $\hat{g}$ such that p_* induces an isometry between $\hat{\mathcal{H}(S)}$ and S, for all elements $S \in \hat{M}$. In other words, we set $\hat{g}(v, w) = g(p_*(v), p_*(w))$, for all $v, w \in \hat{\mathcal{H}}(S)$.

By construction, $\hat{g}$ is a smooth Riemannian metric on $\hat{M}$. For each point $S \in \hat{M}$, the differential p_* sends the orthogonal subspaces $\hat{\mathcal{F}}(S)$ and $\hat{\mathcal{H}}(S)$ to orthogonal subspaces of $T_{p(S)}M$ and the restrictions of p_* to $\hat{\mathcal{F}}(S)$ and to $\hat{\mathcal{H}}(S)$ are 1-Lipschitz. Therefore, the map $p : (\hat{M}, \hat{g}) \to (M, g)$ is 1-Lipschitz.

On the regular part $\hat{M}_0$ the foliation $\hat{\mathcal{F}}$ is a Riemannian foliation with respect to the metric $\hat{g}$. (If $\hat{M}_0$ and M_0 are identified via the diffeomorphism $p : \hat{M}_0 \to M_0$, the metric $\hat{g}$ arises from the metric g by changing g only on $\mathcal{F}$ and by leaving the metric on the normal part unchanged). Since

$\hat{M}_0$ is dense in $\hat{M}$, the foliation $\hat{\mathcal{F}}$ is a Riemannian foliation on the whole manifold $(\hat{M}, \hat{g})$.

By construction, p_* sends horizontal vectors on $\hat{M}$ to horizontal vectors on M of the same length; therefore, p preserves transverse length of curves. Thus $p : (\hat{M}, \hat{\mathcal{F}}) \to (M, \mathcal{F})$ is a geometric resolution.

4.6 Proof of Theorem 1.1

Now we can finish the proof of Theorem 1.1. If $(M, \mathcal{F})$ admits a geometric resolution, then $\mathcal{F}$ is infinitesimally polar, as was shown in Section 3.

Let now $\mathcal{F}$ be infinitesimally polar. Consider the manifold $\hat{M}$ with the foliation $\hat{\mathcal{F}}$ defined in Subsection 4.2 and let $F : \hat{M} \to M$ be the canonical projection p. Let $\hat{g}$ be the Riemannian metric on $\hat{M}$ defined in Subsection 4.5. As we have seen, $\hat{\mathcal{F}}$ is a Riemannian foliation on the Riemannian manifold $(\hat{M}, \hat{g})$ and $F : \hat{M} \to M$ is a geometric resolution.

We have seen in Subsection 4.5, that the map F is 1-Lipschitz. By construction, the leaves of $\hat{\mathcal{F}}$ are preimages of leaves of $\mathcal{F}$, thus p induces a bijection between spaces of leaves. Moreover, by construction, the preimage of a compact subset K on M is a closed subset of a compact subset of the Grassmannian bundle $Gr_k(M)$. Thus the map F is proper.

If M is compact then $\hat{M}$ is compact, since F is proper. Since F is 1-Lipschitz, a ball of radius r around a point $S \in \hat{M}$ is contained in the preimage of the ball of radius r around $F(S)$ in M. If M is complete, the properness of F implies that all balls in $\hat{M}$ are compact. Therefore, $\hat{M}$ is complete in this case.

The objects $(\hat{M}, \hat{\mathcal{F}}, \hat{g})$ are defined only in terms of $M, \mathcal{F}$ and g. Therefore, they are invariant under isometries of $(M, \mathcal{F})$. This proves the statement about Γ-equivariance. The claim about singular Riemannian foliations $\mathcal{F}$ given by orbits of an isometric action of a group G is a direct consequence of the last claim.

Assume now that M and therefore $\hat{M}$ are complete. The notion of the absence of horizontal conjugate point is a transverse notion , i.e., it can be formulated only in terms of local quotients (cf. [LT07a]). Since the transverse geometries of $(M, \mathcal{F})$ and of $(\hat{M}, \hat{\mathcal{F}})$ coincide, due to the definition of a geometric resolution, the singular Riemannian foliation $\mathcal{F}$ has no horizontal conjugate points if and only if the Riemannian foliation $\hat{\mathcal{F}}$ has no horizontal conjugate points.

Identifying the regular part $\hat{M}_0$ with M_0 via F, we see that, by construction, the horizontal distributions of $\mathcal{F}$ with respect to the metrics g and $\hat{g}$ coincide. Thus, one of them is integrable if and only if the other one is integrable. The integrability of the normal distribution on the regular part is equivalent to polarity ([Ale06]). This shows that $\mathcal{F}$ is polar if and only if $\hat{\mathcal{F}}$ is polar.

This finishes the proof of Theorem 1.1.

5. Simplicity in the regular part

We are going to prove Theorem 1.5 in a slightly more general setting that we are going to describe now.

Definition 5.1 *A singular Riemannian foliation on a Riemannian manifold M is full if for each leaf L there is some $\epsilon > 0$ such that $\exp(\epsilon v)$ is defined for each unit vector in the normal bundle L.*

Each singular Riemannian foliation on a complete Riemannian manifold is full. In a full singular Riemannian foliation each pair of leaves is equidistant. If $\mathcal{F}$ is full on M and if $U \subset M$ is an open subset that is a union of leaves of $\mathcal{F}$ then the restriction of $\mathcal{F}$ to U is full again (This follows from [LT07a], Proposition 4.3). Moreover, for each covering N of M the lift of $\mathcal{F}$ to N is full on N.

If $\mathcal{F}$ is a full singular Riemannian foliation on a Riemannian manifold M with all leaves closed, then $M/\mathcal{F}$ is a metric space, with a natural inner metric that has curvature locally bounded below in the sense of Alexandrov. Note that an isometry of such a space is uniquely determined by its restriction to an open subset. Finally, a full Riemannian foliation is simple, i.e., has closed leaves with trivial holonomy, if and only the quotient $M/\mathcal{F}$ is a Riemannian manifold.

Let now $\mathcal{F}$ be a full singular Riemannian foliation on a connected Riemannian manifold M, with $\pi_1(M) = \Gamma$. Let $\tilde{M}$ be the universal covering of M and let $\tilde{\mathcal{F}}$ be the lifted singular Riemannian foliation on $\tilde{M}$. Assume that $\tilde{\mathcal{F}}$ has closed leaves and denote by B the quotient space $\tilde{M}/\tilde{\mathcal{F}}$. The fundamental group Γ acts of $(\tilde{M}, \tilde{\mathcal{F}})$. Thus we get an induced action of Γ on the quotient B. Denote by Γ_0 the kernel of the action of Γ on B, i.e., the set of all elements of Γ that act trivially on B.

Lemma 5.1 *In the notations above let $g \in \Gamma$ be an element. Then the following are equivalent:*

i) *$g \in \Gamma_0$;*

ii) *Each leaf L of $\mathcal{F}$ contains a closed curve whose free homotopy class is the conjugacy class of g;*

iii) *There is a non-empty open subset U in M such that each leaf L of $\mathcal{F}$, which has a non-empty intersection with U, contains a closed curve whose free homotopy class is the conjugacy class of g.*

Proof. Let $\tilde{L}$ be a leaf of $\tilde{\mathcal{F}}$ through a point $y \in \tilde{M}$. Then the translate gy is contained in $\tilde{L}$ if and only if g fixes the point $\tilde{L} \in B$. On the other hand, if gy is contained in $\tilde{L}$ then connecting y and gy by a curve in $\tilde{L}$ one obtains a closed curve in the image L of $\tilde{L}$ in M whose free homotopy class is in the conjugacy class of g. Note that this image L is a leaf of $\mathcal{F}$.

Let L be a leaf in M that contains a closed curve γ whose free homotopy class is in the conjugacy class of g. Then each lifted leaf $\tilde{L}$ of L contains a lift of the curve γ. Thus, in this case, each lift $\tilde{L}$ of the leaf L is fixed by some conjugate of g.

Now the implications $1 \Longrightarrow 2 \Longrightarrow 3$ are clear. Assume 3. Let $\tilde{U}$ be the preimage of U in $\tilde{M}$ and V the projection of $\tilde{U}$ to B. Then V is a non-empty open subset of the quotient B and each point in V is fixed by some conjugate of g. There are only countably many conjugates of g, each of them fixing a closed subset of B. By Baires theorem, at least one conjugate of g fixes a non-empty open subset of V. Since B is an inner metric space with curvature locally bounded from below, g fixes all of B. Therefore, $g \in \Gamma_0$. □

Remark 5.1 *The lemma above is true also in the case of non-closed $\tilde{\mathcal{F}}$, as one sees by localizing the arguments. However, we will use it later only in the case, where $\tilde{\mathcal{F}}$ is a simple Riemannian foliation.*

The following result slightly generalizes Theorem 1.5.

Proposition 5.2 *Let $\mathcal{F}$ be a full singular Riemannian foliation on a simply connected Riemannian manifold M. Let M_0 denote the regular stratum of M and let the Riemannian foliation $\mathcal{F}_0$ be the restriction of $\mathcal{F}$ to M_0. If the lift of $\mathcal{F}_0$ to the universal covering of M_0 is simple then $\mathcal{F}_0$ is a simple Riemannian foliation on M_0.*

Proof. The assumptions and conclusions do not change if one deletes from M all strata of codimension ≥ 3. Thus we may assume that such strata do not exist. Then the complement $\Sigma = M \setminus M_0$ is a disjoint union of closed submanifolds Σ_i of codimension 2.

Choose a point x_i on Σ_i, a small neighborhood P_i of x_i in Σ_i and a small tubular neighborhood U_i of P_i in M. Let $q : U_i \to P_i$ be the foot point projection. The restriction of q to $U_i \setminus P_i$ is a fiber bundle with circles as fibers. By construction, each of these circles is contained in a leaf of $\mathcal{F}$.

On the other hand, all these circles are in the same free homotopy class $[g_i]$ of $U_i \setminus P_i$. Since M is simply connected, the fundamental group Γ of M_0 is generated by conjugates of the elements g_i (i.e., Γ is normally generated by the elements g_i). Due to Lemma 5.1, each of these free homotopy classes acts trivially on the manifold $B = \tilde{M}_0/\tilde{\mathcal{F}}_0$. Thus $\Gamma = \pi_1(M_0)$ acts trivially on B and we get $M_0/\mathcal{F}_0 = B$. Thus $\mathcal{F}_0$ is a simple foliation. □

Remark 5.2 *In fact, the above proof shows that $\tilde{M}_0/\tilde{\mathcal{F}}_0 = M_0/\mathcal{F}_0$ without any assumption on $\mathcal{F}_0$. This fact is essentially contained in [Mol88b], p.213-214.*

Now we are going to provide:

Proof of Theorem 1.6. In [DAM07] it is shown that any Coxeter orbifold is a good orbifold, thus (2) $\Longrightarrow$ (1).

Assume (1). Since B is a Riemannian orbifold, $\mathcal{F}$ is infinitesimally polar ([LT07a]). Consider the geometric resolution $(\hat{M}, \hat{\mathcal{F}})$ of $(M, \mathcal{F})$ defined by Theorem 1.1. Let M_1 be the universal covering of $\hat{M}$ and let $\mathcal{F}_1$ be the lift of $\hat{\mathcal{F}}$ to M_1. Then the quotient $B_1 = M_1/\mathcal{F}_1$ is a Riemannian orbifold, that is an orbifold-covering of B. Since B is a good orbifold, B_1 is a good orbifold as well. But M_1 is simply connected, thus the *orbifold fundamental group* of B_1 is trivial ([Hae88]). Thus B_1 coincides with its universal orbifold covering, hence B_1 is a Riemannian manifold. Therefore, $\mathcal{F}_1$ is a simple Riemannian foliation. From Theorem 1.5 we deduce that the restriction of $\mathcal{F}$ to the set M_0 of regular points is a simple foliation. Hence (1) implies (3).

Assume (3). Then the subset $B_0 = M_0/\mathcal{F}$ of B is a Riemannian manifold. Let $y \in B$ be an arbitrary point. Choose a point x in M on the leaf that corresponds to y. Choose a small distinguished neighborhood U at the point x. Then the restriction of $\mathcal{F}$ to U is given by a (restriction of an) isoparametric foliation on $\mathbf{R}^n$, thus $U/\mathcal{F}$ is a Weyl chamber. The embedding $U \to M$ induces a finite-to-one projection $U/\mathcal{F} \to B$. Moreover, this projection is given by a finite isometric action of a group Γ on $U/\mathcal{F}$ (cf. [LT07a], p.7). The group Γ preserves the set O of regular points of $U/\mathcal{F}$ that is diffeomorphic to an open ball, since $U/\mathcal{F}$ is a Euclidean Weyl chamber. By assumption, the quotient O/Γ is a Riemannian manifold, thus $O \to O/\Gamma$ is a finite covering. But O is contractible and of finite dimension. Hence, $O \to O/\Gamma$ must be a diffeomorphism. Since O is dense in $U/\mathcal{F}$, the action of Γ on $U/\mathcal{F}$ is trivial. Hence a neighborhood of y is diffeomorphic to the Weyl chamber $U/\mathcal{F}$. □

Finally, we are going to deduce Corollary 1.7 from Theorem 1.6.

Proof of Corollary 1.7. The equivalence between (1) and (3) is just the equivalence between (1) and (3) in Theorem 1.6.

If the orbit Gx though x is exceptional then the action of K_x on H_x is a non-trivial action with finite orbits. Thus these orbits cannot coincide with the connected orbits of K_x^0. Hence (2) $\Longrightarrow$ (3).

On the other hand, assume (3). Choose a point $x \in M$ and a distinguished tubular neighborhood U at x. In the notations used in the final part of the proof of Theorem 1.6, the group Γ is just the quotient $\Gamma = K_x/K_x^0$. We have seen that Γ acts trivially on the local quotient $U/\mathcal{F}$, i.e., on the quotient H_x/K_x. Hence the orbits of K_x^0 and of K_x on H_x coincide. Thus (3) $\Longrightarrow$ (2). □

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Alexander Lytchak lytchak@math.uni-bonn.de
A. Lytchak, Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany