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At the beginning of my set-theoretic life I was very interested in cardinal characteristics of the continuum; these are (usually) cardinals of the form

$$\kappa_R := \min\{\, X \subseteq \mathbb{R} \mid \neg\exists y\, \forall x\, (x,y) \in R\,\}$$

i.e. cardinals that answer questions of the form

> What is the smallest number of (real numbers) x that you need so that there is no y with $(x,y) \in R$ for all x?

for some relation R; for example, for the relation $\neq$ this number is the continuum $\mathfrak{c}$ itself, while for the relation $\leq^*$, defined by

$$x \leq^* y \;\Leftrightarrow\; \exists n\, \forall k \geq n\;\, x(k) \leq y(k)$$

the number $\kappa_{\leq^*}$ is usually called $\mathfrak{b}$.

It turns out that many (but not all) ZFC-provable inequalities (such as $\omega_1 \leq \mathfrak{b} \leq \mathfrak{c}$) have trivial (or at least easy) proofs, the consistency of a strict inequality (such as $\mathfrak{b} < \mathfrak{c}$) is often hard, and the consistency of combinations of strict inequalities is in many cases still open.

A standard tool for increasing a cardinal characteristic κ_R is iteration of forcing; the main difficulty in such proofs is to ensure that some other cardinal characteristic κ_S stays small in all intermediate models of the iteration.

In recent times I have become interested in problems in universal algebra that seem to require set-theoretical methods – descriptive set theory, partition theorems, and even forcing; as a typical example, consider the following structure:

> Let ω^ω be the set of all unary functions from ω to ω; with the pointwise order this set is a partial order and even a lattice (any two elements have a least upper and greatest lower bound), and with the operation of composition this set is a monoid (semigroup with neutral element).
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> Let $\mathfrak{S}$ be the family of all subsemigroups which are also ideals (downward closed, and closed under the lub operation).

The set $\mathfrak{S}$, equipped with the subset relation, turns out to be a complete lattice with largest element ω^ω; what more can be said about its structure? One can see that it has many coatoms (lower neighbors of ω^ω); is every element of S other that ω^ω itself contained in a coatom?

The answer is "no", assuming CH (or MA); the proof uses an ultrafilter which describes a sufficiently generic object of a certain σ-closed forcing notion.