Arithmetic and Reduced Powers

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The failure of categoricity for models of arithmetic, in that this stands to refute the idea that we can make our intuitions about arithmetic precise, is a fact of fundamental philosophical importance. One can still try to classify and/or describe these models, but non-standard models of arithmetic are not recursive, meaning that the set of triples belonging to the addition relation (or respectively the multiplication relation) of a (countable) non-standard model is not a recursive set.

Stanley Tennenbaum, who proved their non-recursiveness, also showed that any countable model of arithmetic is embeddable into the reduced power $\mathbb{N}^\omega/F$. The classification project then reduces to that of describing the behavior of (equivalence classes of) functions from $\mathbb{N}$ to $\mathbb{N}$ which happen to belong to models of arithmetic inside that structure. I obtained some results along these lines.

Subsequent research in collaboration with Jouko Väänänen and Saharon Shelah devolved on the question of whether uncountable models of arithmetic were embeddable into $\mathbb{N}^\omega/F$, also the same substituting any regular filter in place of the Frechet filter F, and more generally still the question whether for any first order structure M of any cardinality, every model elementarily equivalent to the reduced power M^λ/D and of cardinality $\leq \lambda^+$ is embeddable into M^λ/D, for D a regular filter.

Over a series of papers, the authors showed that the question as conjectured by Chang and Keisler is independent of ZFC, as is the related question whether $A^\lambda/D \cong B^\lambda/D$ whenever A and B are elementarily equivalent models of size $\leq \lambda^+$ in a language $\leq \lambda$ and D is a regular filter. This was done by isolating a principle equivalent to the original conjectures, namely a finitary square principle $\Box^{fin}_{\lambda,D}$, a variant of $\Box_\lambda$. Questions to be taken up in subsequent research are whether proving estimates for the consistency strength of $\neg\Box^{fin}_{\lambda,D}$ can be obtained, and whether $\Box^{fin}_{\lambda,D}$ has a similar relation as other square principles to axioms like PFA.