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3. Trees and the EF-game

In this lecture we establish the basic connections between trees and Ehrenfeucht-Fraïssé games of length ω_1. In a sense we try to generalize the concept of a Scott rank from countable models to uncountable ones.

To get a feeling about transfinite Ehrenfeucht-Fraïssé games, let us consider the following:

Example (Tuuri) Let $\mathfrak{A} = (A, E)$ be a model with an equivalence relation E with $\aleph_1$ classes of size $\aleph_1$[1] and also $\aleph_1$ classes of size $\aleph_0$, we call them "small". Respectively, let $\mathfrak{A}' = (A', E')$ be a model with an equivalence relation E' with $\aleph_0$ small classes and $\aleph_1$ big classes. Clearly, I wins the Ehrenfeucht-Fraïssé game if he can play $\omega \cdot 2 + 1$ moves. However II can avoid loss for $\omega \cdot 2$ moves as follows:

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[1] we call them "big"

During the first ω moves she follows some preassigned bijections between elements in the small classes, matching a "small" element with a "small" element and a "big" element with a "big" element. After these ω moves there are ω moves left. Now she cannot match every big element with a big element if I has played well during the first ω moves. However, this does not matter, for with ω moves I cannot call II's bluff.

The above example, which can of course be vastly generalized, demonstrates the power of a transfinite Ehrenfeucht-Fraïssé game in revealing infinitary differences between uncountable models. Note that for the above models $\mathfrak{A}$ we have trivially $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{A}'$ but not $\mathfrak{A} \equiv_{\infty\omega_1} \mathfrak{A}'$.

The following important example is due to M. Morley:

Example Suppose $A \subseteq \omega_1$. If $\alpha < \omega_1$, let

$$r_\alpha = \begin{cases} 1+\eta & \text{if } \alpha \in A \\ \eta & \text{if } \alpha \notin A \end{cases}$$

where η is the order-type of the rationals.
Let

$$\Phi(A) = \sum_{\alpha<\omega_1} r_\alpha .$$

Let $\Phi_\delta(A) = \sum_{\alpha<\delta} r_\alpha$.

Then (Conway) $\Phi(A) \cong \Phi(B)$ iff $A \triangle B$ is non-stationary.

To prove this suppose first $A \triangle B$ is stationary. Suppose $f : \Phi(A) \cong \Phi(B)$. Let $M \prec H(\lambda)$, λ large, be countable so that $A, B, \omega_1, \Phi(A), \Phi(B), f \in M$. Let $\delta = M \cap \omega_1$. W.l.o.g. $\delta \in A \triangle B$. Clearly, $f \restriction \Phi_\delta(A) : \Phi_\delta(A) \cong \Phi_\delta(B)$. However, $\delta \in A \triangle B$, say $\delta \in A \setminus B$. Let a be the first element of r_δ in $\Phi(A)$. Now a is the supremum of $\Phi_\delta(A)$ in $\Phi(A)$, but $\Phi_\delta(B)$ has no supremum in $\Phi(B)$, so $f(a)$ cannot be such.

This argument actually shows more than just $\Phi(A) \not\cong \Phi(B)$. It shows that II cannot have a winning strategy in an EF-game of length $\omega+2$. Namely, we could pick the above set $M \prec H(\lambda)$ to be closed under the first ω moves of the winning strategy of II. Then during the first ω moves of the game I would enumerate $\Phi_\delta(A)$ and $\Phi_\delta(B)$, and II would be trapped during move number $\omega+1$.

So

II has a winning strategy of length $\omega+2$

$\Rightarrow$ $A \triangle B$ is non-stationary.

On the other hand, suppose $A \triangle B$ is disjoint from a club $C \subseteq \omega_1$. Then II has the following winning strategy even in the EF-game of length ω_1: During the game II maintains an isomorphism f between initial segments $\Phi_\delta(A)$ and $\Phi_\delta(B)$, where $\delta \in C$. If I plays inside $\Phi_\delta(A) \cup \Phi_\delta(B)$, II uses f. If I goes beyond $\Phi_\delta(A) \cup \Phi_\delta(B)$, then II picks a large enough $\delta' > \delta$ from C and extends f to $f: \Phi_{\delta'}(A) \cong \Phi_{\delta'}(B)$. The extension exists because $\Phi_{\delta'}(A) \setminus \Phi_\delta(A)$ and $\Phi_{\delta'}(B) \setminus \Phi_\delta(B)$ are countable dense linear orderings with first (if $\delta \notin A$, otherwise w/o first) but no last point. Since C is closed, II can follow this strategy over limits, too.

This argument shows that for all α

I has a winning strategy of length α

$\Rightarrow$ $A \triangle B$ is stationary.

As a special case we get the result that if A is bistationary then the EF-game of length $\omega+2$ between $\Phi(A)$ and $\Phi(-A)$ is non-determined.

Note that if $A \triangle B$ is non-stationary then $\Phi(A) \equiv_{\infty\omega_1} \Phi(B)$. This shows that $L_{\infty\omega_1}$-equivalent models of cardinality $\aleph_1$ need not be isomorphic.

Needless to say, constructions like $\Phi(A)$ are useful in larger cardinals, too, but then one cannot take for granted that intervals with first but not last element are isomorphic, at least density alone does not guarantee this.

After these preliminary remarks, let us define the Ehrenfeucht-Fraisse game $EF_P(\mathfrak{A},\mathfrak{B})$ between $\mathfrak{A}$ and $\mathfrak{B}$ (the domains of which are assumed to be disjoint) with a poset P as the "clock":

I	t_0, x_0	t_1, x_1	$\cdots$	t_ω, x_ω	$\cdots$
II	y_0	y_1	$\cdots$	y_ω	$\cdots$

The moves

I	x_0	x_1	$\cdots$	x_ω	$\cdots$
II	y_0	y_1	$\cdots$	y_ω	$\cdots$

are like in an ordinary (transfinite) Ehrenfeucht-Fraisse game. Additionally, I has to go up the po-set P:

$$t_0 < t_1 < \cdots < t_\omega < \cdots$$

The player who cannot move loses.

As for trees, we define $P \leq P'$ if there is an order-preserving $f: P \to P'$ i.e.

$$x <_P y \Rightarrow fx <_{P'} fy.$$

Likewise σP is the tree of chains in P and $P \ll P'$ if $\sigma P \leq P'$.

The following lemma shows that $\leq$ is exactly what is needed for comparing games:

<u>Lemma</u>[1] 1) If $\mathrm{II} \uparrow EF_P(\mathfrak{A}, \mathfrak{B})$ and $P \leq P'$, then $\mathrm{II} \uparrow EF_{P'}(\mathfrak{A}, \mathfrak{B})$.

2) If $\mathrm{I} \uparrow EF_P(\mathfrak{A}, \mathfrak{B})$ and $P' \leq P$, then $\mathrm{I} \uparrow EF_{P'}(\mathfrak{A}, \mathfrak{B})$

3) If $\mathrm{II} \uparrow EF_P(\mathfrak{A}, \mathfrak{B})$ and $\mathrm{I} \uparrow EF_{P'}(\mathfrak{A}, \mathfrak{B})$, then $P \ll P'$.

Since (1) and (2) are trivial, we prove only (3)!

1) $\mathrm{I} \uparrow EF_P$ means I has a winning strategy in EF_P. Similarly for $\mathrm{II} \uparrow EF_P$.

Let τ be the winning strategy of II in $EF_{\mathcal{P}}$, and σ the winning strategy of I in $EF_{\mathcal{P}'}$. In order to show that $\mathcal{P} << \mathcal{P}'$, we describe a winning strategy of I in $G(\mathcal{P}', \mathcal{P})$. Suppose

$$t'_0 < t'_1 < \ldots < t'_\nu < \ldots \quad (\nu < \alpha)$$

has been played in $\mathcal{P}'$ by I in $G(\mathcal{P}', \mathcal{P})$, and

$$t_0 < t_1 < \ldots < t_\nu < \ldots \quad (\nu < \alpha)$$

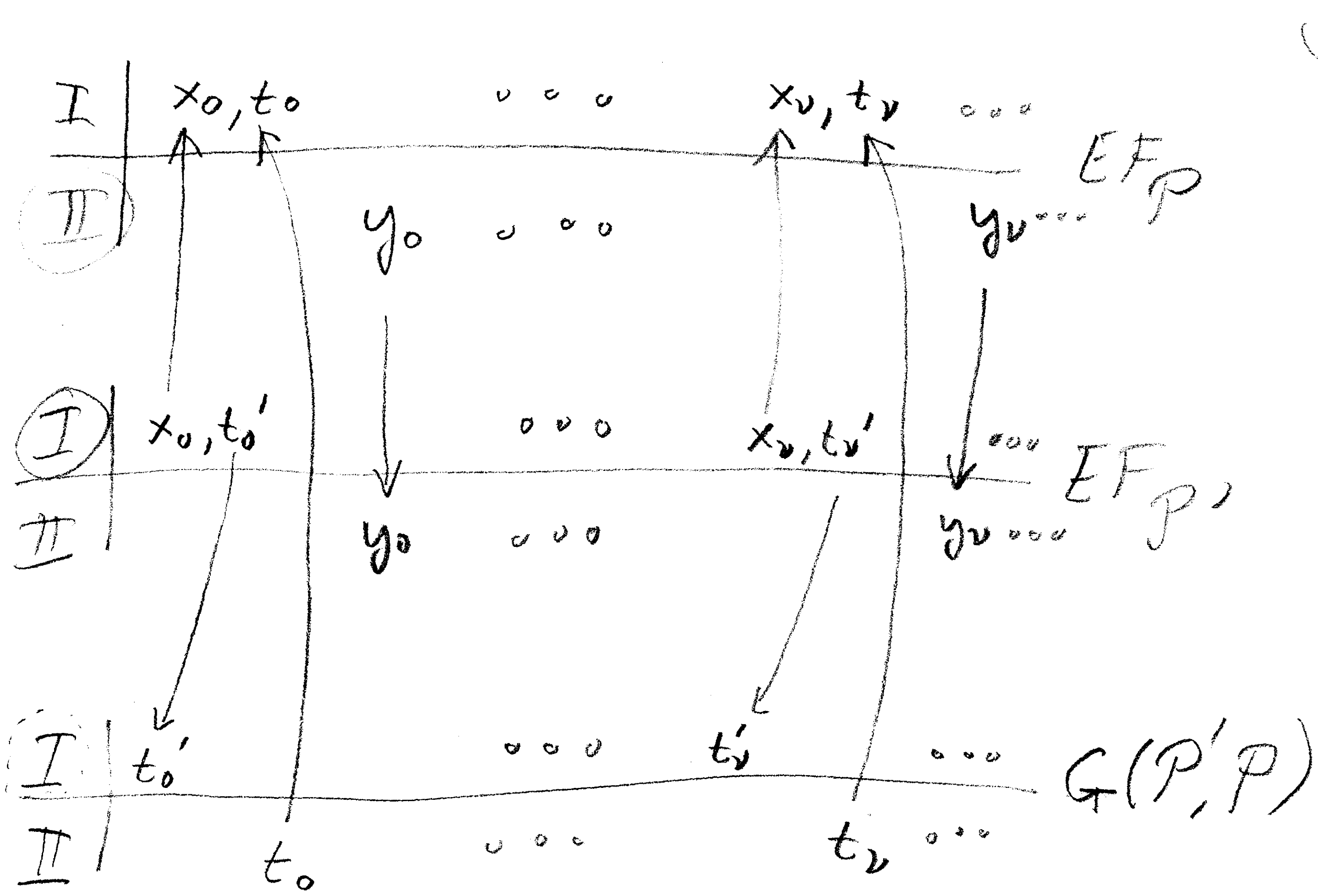
has been played in $\mathcal{P}$ by II. During the game also elements x_ν, y_ν $(\nu < \alpha)$ have been played so that

$$\begin{cases} (x_\nu, t'_\nu) = \sigma(\langle y_\xi ; \xi < \nu \rangle) \\ y_\nu = \tau(\langle (x_\xi, t_\xi) ; \xi \leq \nu \rangle). \end{cases}$$

In the beginning $(x_0, t'_0) = \sigma(\emptyset)$ so I plays t'_0 in $G(\mathcal{P}', \mathcal{P})$. When II responds with t_0, I can let $y_0 = \tau((x_0, t_0))$ and compute

$$(x_1, t'_1) = \sigma(\langle y_0 \rangle),$$

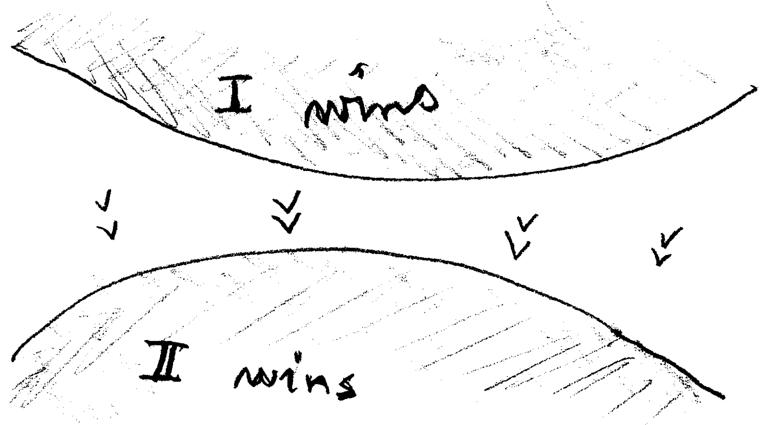
etc, etc. The following picture clarifies the strategy:



$\square$

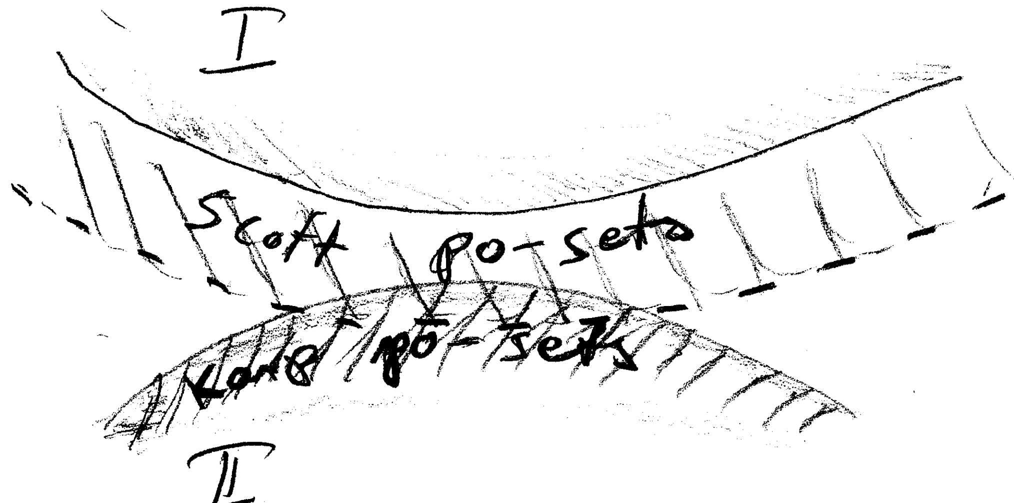
The Lemma demonstrates that the po-sets P for which II wins EF_P and for which I wins are separated by $\ll$, in effect by the σ-operation. The Lemma also demonstrates that the class of posets P for which II wins EF_P is $\leq$-closed downwards. It is easy to see that it is also closed under disjoint union (= supremum under $\leq$). Respectively, the class of po-sets P for which I wins EF_P

is closed upwards with respect to $\leq$, and it is clearly closed under $\otimes$ (i.e. infimum with respect to $\leq$).



To understand better the situation we pay more attention to the *boundary* region.

A po-set P is a *Karp po-set* of $\mathfrak{A}$ *and* $\mathfrak{B}$ if II has a winning strategy in $EF_P(\mathfrak{A},\mathfrak{B})$ but not in $EF_{\sigma P}(\mathfrak{A},\mathfrak{B})$. It is a *Scott po-set* of $\mathfrak{A}$ *and* $\mathfrak{B}$ if I has a winning strategy in $EF_{\sigma P}(\mathfrak{A},\mathfrak{B})$ but not in $EF_P(\mathfrak{A},\mathfrak{B})$. Note that if α is the Scott watershed of $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathcal{B}_\alpha$ is both a Scott and a Karp po-set of $\mathfrak{A}$ and $\mathfrak{B}$. Naturally, we are now more interested in the situation when the Scott watershed does not exist (i.e. $\mathfrak{A} \simeq_p \mathfrak{B}$). Then the boundary between the Karp po-sets and Scott posets exemplifies what would be the Scott watershed if such existed.



Lemma (1) Karp posets always exist, if $\mathfrak{A} \not\cong \mathfrak{B}$.
(2) Scott posets always exist, if $\mathfrak{A} \not\cong \mathfrak{B}$

Proof (1) Let K be the po-set of all winning strategies of II in EF-games of a fixed length α, where $\alpha \in On$. Since we assume $\mathfrak{A} \not\cong \mathfrak{B}$, there is an upper bound (card(A) + card(B)) for such α. We order K by canonical and-extension of strategies.

<u>II wins EF_K</u>: While EF_K is played player I reveals more and more of the possible winning strategies of II and II can take advantage of them. Thus if I has played

$$(x_0, t_0), \ldots, (x_\nu, t_\nu) \quad (\nu \leq \alpha)$$

player II can play

$$y_\alpha = t_\alpha(\langle x_\xi : \xi \leq \alpha \rangle).$$

Since $t_0, \ldots, t_\nu, \ldots, t_\alpha$ is an increasing sequence of winning strategies of II, II cannot lose.

<u>II does not win $EF_{\sigma K}$</u>: Note that we do not try to show that I has a winning strategy in $EF_{\sigma K}$! Suppose II has a winning strategy τ in $EF_{\sigma K}$. It follows that $\mathfrak{A} \cong \mathfrak{B}$, for

II has a winning strategy τ_α in the EF-game of length α for all α: In the game of length α she uses τ and pretends that I is playing on round β

$$(x_\beta, \langle \tau_\gamma : \gamma < \beta \rangle).$$

Since τ_γ is defined for all $\gamma < \alpha$, this is well-defined. We get $\mathfrak{A} \cong \mathfrak{B}$, a contradiction

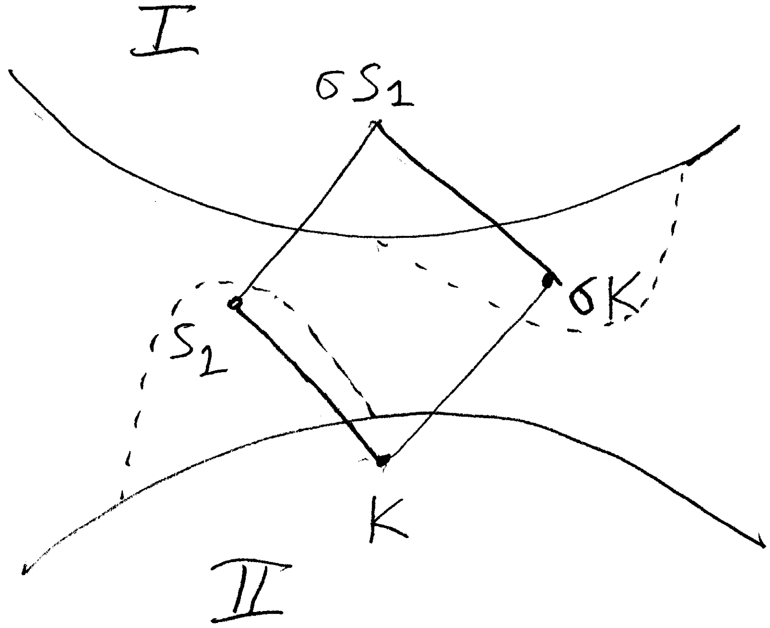
(ii) Suppose $\mathfrak{A} \not\cong \mathfrak{B}$. Let α be such that I has a winning strategy σ in the EF-game of length α (E.g. $\alpha = |A| + |B|$) Let S_0 be the po-set of all plays in which I has used σ and II has not lost yet. We order S_0 by end-extension.

<u>I wins $EF_{\sigma S_0}$:</u> Player I wins by following σ and submitting the sequence of all previous plays as the "clock"-move. He can follow this strategy without end and hence wins.

Let S_1 be $\ll$-minimal such that I wins $EF_{\sigma S_1}$.

<u>I does not win EF_{S_1}</u>. Suppose I has a winning strategy σ_0 in EF_{S_1}. Let S_2 be the po-set of plays in EF_{S_1} when I uses σ_0. As above, I wins $EF_{\sigma S_2}$. This contradicts the minimality of S_1 as clearly $\sigma S_2 \leq S_1$.

□



Note that since II wins EF_K and I wins $EF_{\sigma S_2}$, we have $\sigma K \leq \sigma S_1$. In fact, $K \leq S_1$.

□

Lemma If T is <u>any</u> Karp tree, then $T \leq K$, where K is as above.

Proof Suppose II has a winning strategy τ in EF_T. Let $t \in T$ have <u>rank</u> α. Then we get from T a winning strategy τ^* of II in an EF-game of length α by simply letting I play one by one the predecessors of t. The mapping $\tau \mapsto \tau^*$ shows that $T \leq K$. $\square$

Thus we have a largest Karp tree K. There are models with two Scott trees the inf of which is not a Scott tree, so there need not be any smallest Scott tree. Of course, there are always $\ll$-minimal Scott-trees, since $\ll$ is well-founded.

Note that if P is both a Karp- and a Scott poset of $\mathfrak{A}$ and $\mathfrak{B}$, then $EF_{P'}(\mathfrak{A},\mathfrak{B})$ is determined for all P' which are comparable with P

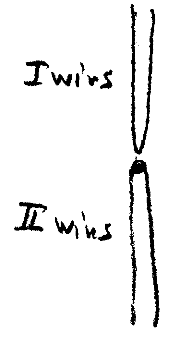
$P' \leq P \implies$ II wins $EF_{P'}$

$P \ll P' \implies$ I wins $EF_{P'}$.

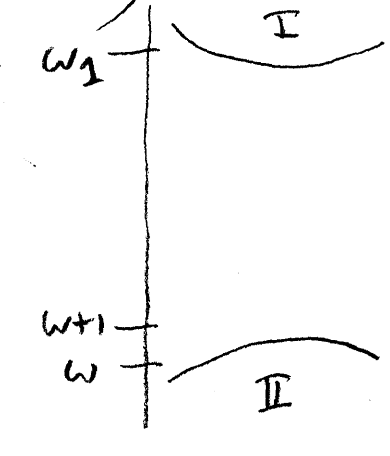
For posets P' such that $P' \not\leq P$ and $P \not\leq P'$ we still know nothing about the determinacy of $EF_{P'}$. This shows the importance of bottlenecks.

Examples:

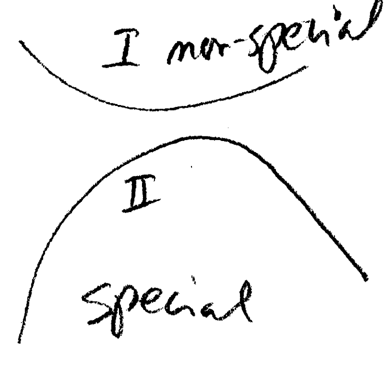
1) $\mathfrak{A} \simeq_P \mathfrak{B}$



2) $\Phi(A), \Phi(-A)$



3) Next theorem



<u>Theorem</u> (Tuuri) 1) There are models $\mathfrak{A}$ and $\mathfrak{B}$ so that $(\mathbb{Q},<)$ is both a Karp-po-set and a Scott-po-set of $\mathfrak{A}$ and $\mathfrak{B}$.
2) There are models $\mathfrak{A}$ and $\mathfrak{B}$ such that $(\mathbb{R},<)$ is both a Karp-po-set and a Scott-po-set of $\mathfrak{A}$ and $\mathfrak{B}$.

Let ρ_0 be the least κ (if any) such that if T is any non-special tree, then there is a non-special subtree T_0 of T of cardinality $<\kappa$. The statement $\rho_0=\aleph_2$ is known as the Rado Conjecture. It is consistent relative to the consistency of a supercompact cardinal ([52, 54]), and Chang's Conjecture.

Let κ_0 be the least κ (if any) such that if $\mathfrak{A}$ and $\mathfrak{B}$ are models of card $\leq 2^{\omega}$ and T is a Karp-tree of $\mathfrak{A}$ and $\mathfrak{B}$, then there is a subtree T_0 of T such that $|T_0|<\kappa$

and T_0 is a Karp-tree of $\mathfrak{A}$ and $\mathfrak{B}$. Note that if "Karp" is replaced by "Scott", such κ would always exist, and $\kappa \leq (2^\omega)^+$.

Hurokonen [11] showed that if κ is strongly compact, then $\hbar_0$ exists and $\hbar_0 \leq \kappa$. The numbers $\hbar_0$ and μ_0 have a simple relationship:

$$\mu_0 \leq \hbar_0 .$$

For, suppose T is non-special, then $T_0 = T \otimes \sigma\mathbb{Q}$ is non-special and $\leq (\mathbb{R}, <)$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in the above theorem, part (2). Then T_0 is a Karp-tree of $\mathfrak{A}$ and $\mathfrak{B}$. Let T_1 be a subtree of T_0 of size $< \hbar_0$ such that T_1 is a Karp tree. Then $\mathrm{II} \uparrow EF_{\sigma T_1}(\mathfrak{A}, \mathfrak{B})$, whence $\sigma T_1 \not\leq (\mathbb{R}, <)$. Thus T_1 is non-special.

It follows that $\delta_0 = \aleph_2$ implies Radó's Conjecture. For more on δ_0, see [57] (19

T is a universal Karp-tree of $\mathcal{A}$ if for all $\mathcal{B}$ of the same cardinality as $\mathcal{A}$, $\mathrm{II} \uparrow EF_T(\mathcal{A},\mathcal{B})$ implies $\mathcal{A} \cong \mathcal{B}$. Respectively, T is a universal Scott-tree of $\mathcal{A}$ if for all $\mathcal{B}$ of the same cardinality as $\mathcal{A}$, $\mathcal{A} \not\cong \mathcal{B}$ implies $\mathrm{I} \uparrow EF_T(\mathcal{A},\mathcal{B})$. (Universal Scott is, of course, universal Karp)

Countable models have universal Karp- and Scott-trees, even well-founded ones. For uncountable models the situation is more complex. Let us first note that if $2^{\aleph_0} = 2^{\aleph_1}$, then every model of size $\aleph_1$ has a universal Karp tree of cardinality $2^{\aleph_0}$.

On the other hand, if CH is assumed, then there are models of size $\aleph_1$ without a universal Karp-tree of size $\aleph_1$ ([25]).

(CH) A model of size $\aleph_1$ has a universal Scott tree of size $\aleph_1$ if and only if the orbit of $\mathfrak{A}$ is $\underset{\sim}{\Delta}^1_1$. For suppose such a tree T exists. Then we can $\underset{\sim}{\Sigma}^1_1$-define the class $\{\mathfrak{B} : \mathfrak{B} \not\cong \mathfrak{A}\}$ by reducing it to the existence of a winning strategy of I in $EF_T(\mathfrak{A}, \mathfrak{B})$. Conversely, suppose the orbit of $\mathfrak{A}$ is $\underset{\sim}{\Delta}^1_1$. By the Covering Lemma of $\omega_1^{\omega_1}$ we get a tree T of size $\aleph_1$ (by CH) such that T is a universal Scott tree of $\mathfrak{A}$. Mekler and Shelah [33] showed that the free Abelian group $F(\omega_1)$ on $\aleph_1$ generators has a universal Scott-tree in a model of CH. In fact, (assuming CH) $F(\omega_1)$ has a universal Scott-tree, i.e. the orbit of $F(\omega_1)$ is $\underset{\sim}{\Delta}^1_1$, if and only if there is a single tree T of cardinality $\aleph_1$ which has no uncountable branches, but is above (in the sense of $\leq$) of every $T(A)$, $A \subseteq \omega_1$ co-stationary.

In recent work of Shelah, he shows that there are models of size $\aleph_1$ such that the models are non-isomorphic, but II has a winning strategy in EF_α for all $\alpha < \omega_1$. This was open for a long time.

Hyttinen, Tuuri and Shelah relate stability theoretic properties of models to the existence of universal Karp and Scott trees, see bibliography.

The Main Gap of Shelah is roughly the following major result: If T is a countable complete first order theory, then exactly one of the following cases occurs:

Case 1 (Non-structure Case) T has a large number of very similar non-isomorphic models in each uncountable cardinality.

Case 1 (Structure Case) In each uncountable cardinality the models of the theory can be characterized in terms of dimension-like invariants, up to isomorphism.

The long EF-games can be used to strengthen the non-structure case. The somewhat vague "very similar" is taken to mean $\mathrm{II} \uparrow EF_{P}(\mathcal{A}, \mathcal{B})$ for a large class of P. In this way the "clocks" P in a sense grade the levels of non-structure.