2. Trees, po-sets and their ordering

We observed above that the concept of a tree (or more generally, a po-set) without uncountable chains is crucially important in the study of the topology of models of size $\aleph_1$. Likewise, the order $T \le T'$ (there is $f: T \to T'$ such that $x <_T y \Rightarrow f(x) <_{T'} f(y)$), or $P \le P'$ for po-sets P, P', is is in a major role.

We write $T < T'$ if $T \le T'$ and $T' \not\le T$[1]. There are natural operations $T \oplus T'$ (disjoint sum) and $T \otimes T'$ (level-by-level product) for trees. Also, we can define $T \cdot T'$ as the product in

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[1] $T \equiv T'$ means $T \le T'$ & $T' \le T$.

which each node of T' is replaced by a copy of T.

Example If α is an ordinal, let B_α be the tree of descending chains in α, ordered by end-extension. Then

$$\alpha < \beta \Leftrightarrow B_\alpha < B_\beta$$

$$B_{\alpha+\beta} \equiv B_\alpha \oplus B_\beta$$

$$B_{\alpha\cdot\beta} \equiv B_\alpha \cdot B_\beta$$

Example (Todorcevic [53]) If $A \subseteq \omega_1$, the tree $T(A)$ is defined as the set of closed bounded sequences of elements of A, ordered by end-extension. If A and B are disjoint stationary sets, then

$$T(A) \nleq T(B).$$

Proof Suppose $f : T(A) \to T(B)$ is order-preserving. Let $M \prec H(\mu)$, μ large, such that $|M| = \aleph_0$ and $A, f, B \in M$. Let $\delta = M \cap \omega_1$. We can choose M so that $\delta \in A$. Let $\delta_n \uparrow \delta$.

Since $M \prec H(\mu)$, there are s_n in $T(A)$ such that $s_0 < s_1 < \dots$, $ht(s_n) = \delta_n$, and $\sup_n \max(s_n) = \delta$. W.l.o.g., f is level-preserving. So $ht(f(s_n)) = \delta_n$. Let $s = (\bigcup_n s_n) \cup \{\delta\}$. Since $\delta \in A$, $s \in T(A)$. So $f(s) \in T(B)$. But $\delta \notin B$, ↯. $\square$

There is a stronger result about the trees $T(A)$, due also to Todorcevic [53]:

<u>Theorem</u> (i) A is non-stationary $\Leftrightarrow$ $T(A)$ is special

(ii) $A \triangle B$ is non-stationary $\Leftrightarrow$ $T(A) \otimes T(B)$ is special.

$(\, T \otimes T' = \{(s,s') \in T \times T' \,;\, ht_T(s) = ht_{T'}(s')\}\,)$

<u>Proof</u> (i) Suppose $C \cap A = \emptyset$, C club. For $s \in T(A)$ let $f(s)$ be maximal $s' < s$ such that $\max(s') < \delta$, where δ is the largest $\delta \in C$ such that $\delta < \max(s)$. Note that $f(s)$ exists because s is closed. Todorcevic [53] proves the following <u>Pressing Down Theorem for trees</u>: If $f : T \to T$ satisfies $f(s) < s$ for all s, and T is non-special, then there is a non-special $T_0 \subseteq T$ such that $f \restriction T_0$ is constant.

So if $T(A)$ is non-special, there is a non-special $T_0 \subseteq T(A)$ such that $f \restriction T_0$ is constant s_0. Let $s_1 \in T_0$, $s_1 > s_0$, so that s_1 has extensions of arbitrary high height in T_0. Let $\delta \in C$ be minimal so that $\delta > \max(s_1)$. Let $s_2 \in T_0$ so that $s_2 > s_1$, $\max(s_2) > \delta$ and $\max(s_2)$ is minimal, then $f(s_2) = s_1$. On the other hand, $f(s_2) = s_0$, ※.

For the converse, suppose $T(A)$ is special, $T(A) = \bigcup_n A_n$, where each A_n is an antichain. Suppose A is stationary. Let $M \prec H(\lambda)$, λ large, such that $\forall n\,(A_n \in M)$, $T(A), A \in M$ and $\delta = M \cap \omega_1 \in A$. Let $\delta_n \uparrow \delta$. We construct

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$t_0 < t_1 < \cdots$ in $T(A)$ as follows:
Let $t_0 \in \mathcal{A}_0 \cap M$. If there is $t_{n+1} > t_n$ such that $\max(t_{n+1}) \geq \delta_n$ and $t_{n+1} \in \mathcal{A}_{n+1}$, we let t_{n+1} be such. Otherwise $t_{n+1} > t_n$ is arbitrary with $\max(t_{n+1}) \geq \delta_n$. Let $t^* = (\bigcup_n t_n) \cup \{\delta\}$. Then $t^* \in T(A)$.
Hence $t^* \in \mathcal{A}_n$ some n. By construction, $t_n \in \mathcal{A}_n$ and $t_n < t^*$, a contradiction

(ii) Suppose $C \cap (A \triangle B) = \emptyset$, C club, but $T(A) \otimes T(B)$ is non-special.
If $t = (s, s')$ is in $T(A) \otimes T(B)$, let δ be the largest $\delta \in C$ such that $\delta \leq \max(s)$ and $\delta \leq \max(s')$. Note that it is not possible that both $\delta = \max(s)$

and $\delta = \max(s')$, as $C \cap A \cap B = \emptyset$.
Say, $\delta < \max(s)$. Let (s_0, s_0') be maximal $< (s, s')$ such that $\max(s_0)$, $\max(s_0') \leq \delta$. Let $f((s,s')) = (s_0, s_0')$.
By Pressing Down there is non-special $T_0 \subseteq T$ such that $f \restriction T_0$ is constant.
We get a contradiction as above.

Conversely, suppose $T(A) \otimes T(B)$ is special, but $A \triangle B$ is stationary. One derives a contradiction easily, as above. $\square$

Todorcevic [53] goes further into the structure of trees with no uncountable branches by defining the ideal NS_T for any tree T as the set of all $E \subseteq \omega_1$ for which there is a regressive $f: T \upharpoonright E \to T$ such that $f^{-1}(s)$ is special for all $s \in T$. He proves that NS_T is a normal ideal which is trivial iff T is special. Moreover, $T \leq U$ implies $NS_U \subseteq NS_T$ and $NS_T \cup NS_U \subseteq NS_{T \otimes U}$.

From the trees $T(A)$ we get an antichain of size $2^{\aleph_1}$ of trees of size 2^{ω} without uncountable branches.

Better:

<u>Theorem</u> (Todorcevic [56]) There is an antichain of size $2^{\aleph_1}$ of Aronszajn trees. There are chains of all countable lengths as well.

Todorcevic–V. [57] show that with $\diamondsuit$ we can have Souslin trees instead of Aronszajn trees in the above theorem.

Lemma (Kurepa 1956) $T < \sigma T$ (More generally, $\mathbb{P} < \sigma \mathbb{P}$ if $\mathbb{P}$ is a po-set)

Proof Trivially $T \leq \sigma T$. Suppose f is order-preserving $\sigma T \to T$. Let

$$t_\alpha = f(\{f(t_\beta) : \beta < \alpha\}).$$

Now $\{t_\alpha : \alpha \in On\}$ is a strictly increasing proper class of elements of T, ∇ $\square$

<u>Lemma</u> (Hyttinen–V. [26]) If we write

$$T \ll T' \text{ iff } \sigma T \leq T'$$

then $\ll$ is a well-founded quasi-order. (This is true also for po-sets).

<u>Proof</u> Suppose $T_0 \gg T_1 \gg \ldots$ Let $f_n : \sigma T_{n+1} \to T_n$ be order-preserving. Let $t_n^0 = \emptyset$ in σT_n and

$$\begin{cases} t_n^{\alpha+1} = t_n^{\alpha} \cup \{f_n(t_{n+1}^{\alpha})\} \\ t_n^{\nu} = \bigcup_{\alpha<\nu} t_n^{\alpha}, \quad \nu = \cup\nu \end{cases}$$

Again, $\{t_0^{\alpha} : \alpha \in On\}$ is a strictly increasing proper class in T_0, $\nexists$. $\square$

One can define the orderings $T \leq T'$ and $T << T'$ also via a <u>Comparison Game</u> $G(T,T')$. In this game player I goes up T and II goes up T'. The one who cannot move loses.

<u>Fact</u> (1) $T \leq T'$ iff II has a winning strategy in $G(T,T')$

(2) $T' << T$ iff I has a winning strategy in $G(T,T')$.

<u>Proof</u> (1) If $T \leq T'$ via f, then II wins by using f. Conversely, if II has a winning strategy τ, an order-preserving $f: T \to T'$ can be defined by letting I play predecessors of each

$t \in T$ separately.

(2) If $\sigma T' \leq T$ via f, then I wins by playing

$$t_\alpha' = f(\langle t_\alpha : \alpha < \delta \rangle)$$

Where $\langle t_\alpha : \alpha < \delta \rangle$ is the sequence of previous moves of II in T. Conversely, if T has a winning strategy, we can define an order-preserving $f : \sigma T' \to T$ by letting

$$f(\langle t_\alpha' : \alpha < \delta \rangle) = t$$

where $\langle t_\alpha' : \alpha < \delta \rangle \in \sigma T'$ and t is the <u>next</u> move of I after II has played $\langle t_\alpha' : \alpha < \delta \rangle$ and I has played f on shorter sequences.

The fact that there are T, T' such that neither $T \leq T'$ nor $T' \ll T$, shows that $G(T, T')$ may be <u>non-determined</u>.

Further properties of $\leq$ and $\ll$:

<u>Fact</u> (i) There is no T' such that
$T \ll T' \ll \sigma T$.
(i.e. σT is a kind of "successor" of T. Of course $\sigma B_\alpha \equiv B_{\alpha+1}$.)

(ii) $T_1 \otimes T_2 = \inf(T_1, T_2)$

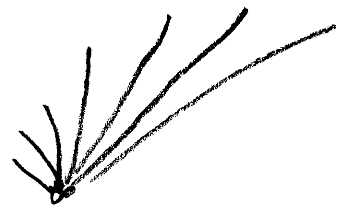
(iii) $T_1 \oplus T_2 = \sup(T_1, T_2)$

[$T_1 \oplus T_2$ is the disjoint sum of T_1 and T_2]

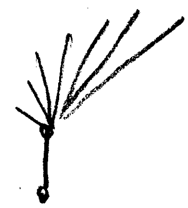
(iv) If T and T' have no infinite branches, then $T \leq T'$ or $T' \leq T$ and $T < T'$ iff $T \ll T$!

<u>Note</u>: The above (i) is true for po-sets, as well.

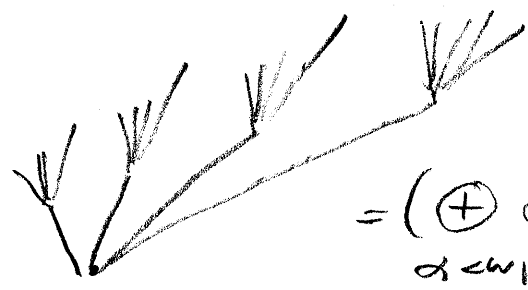
Given the complexity of the structure of trees, it is natural to look for <u>canonical trees</u> i.e. trees that manifest regularity and that are somehow natural. The trees B_α form a proper class which is well-ordered by $\leq$. After these come the trees of height ω with an infinite branch, all equivalent. All trees of successor height $\alpha+1$ are equivalent, so the next interesting case is the class of trees of height $\omega \cdot 2$ with no branch of length $\omega \cdot 2$. In size $\aleph_1$ there are antichains of size $2^{\aleph_1}$ and chains of length ω_1 among these. The same happens at every limit ordinal $< \omega_1$. Finally we come [57] to trees of height ω_1 with no uncountable branches. The simplest of these is the *fan*

$F =$  $= \bigoplus_{\alpha<\omega_1} \alpha$

of single branches of all lengths $< \omega_1$ joined at the root. Note that



is strictly bigger than F, as is $F \cdot 2$

$F \cdot 2 =$  $= \left(\bigoplus_{\alpha<\omega_1} \alpha\right) \cdot 2$

We enter the phenomenon of *persistency*, a concept introduced by T. Huuskonen [10,11].

Before we enter persistency, let us recall some equivalent conditions of speciality of trees. A tree is _special_ if it satisfies one of the following equivalent conditions:

(1) There is $f : T \to \omega$ such that $t < t'$ implies $f(t) \neq f(t')$ (i.e. T is a countable union of antichains).

(2) There is $f : T \to \mathbb{Q}$ such that $t < t'$ implies $f(t) < f(t')$.

(3) II has a winning strategy in the *specializing game* in which I plays elements from T in ascending order and II responds with different element of ω.

(4) $T \leq T_{\aleph_0}$ where $T_{\aleph_0}$ is the tree of sequences of distinct elements of ω, of successor length and so that each sequence omits infinitely many new. The order is end-extension.

<u>Proof</u> (1) $\to$ (2) If $T = \bigcup_n A_n$, A_n anti-chain, then $f : T \to \mathbb{Q}$ is constructed so that the range of $f \upharpoonright \bigcup_{m \leq n} A_m$ is always finite.

(2) $\to$ (4) $\to$ (3) trivial.

(3) $\to$ (1) If $t \in T$ and $(t_\alpha)_{\alpha \leq \delta}$ is the sequence of predecessors of t in T, $t_\delta = t$, let I play $(t_\alpha)_{\alpha \leq \delta}$ in the specializing game. Let the responses of II be $(r_\alpha)_{\alpha \leq \delta}$. We let $f(t) = r_\delta$.

$\square$

We now define persistency, introduced by Hinshman [10]. It resembles speciality very much.

A tree T of height ω_1 is <u>persistent</u> if it satisfies one of the following equivalent conditions:

(1) T has a subtree T' such that every $t \in T'$ has extensions of all heights $< \omega_1$ in T'.

(2) $T_p \leq T$, where $T_p = (\bigoplus_{\alpha < \omega_1} \alpha) \cdot \omega$ (i.e. the result of replacing every maximal node of the fan $\bigoplus_{\alpha < \omega_1} \alpha$ by a copy of the fan, and repeating this ω times)

(3) II has a winning strategy in the <u>persistency game</u> in which I plays ordinals $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ $(n < \omega)$ below ω_1 and II responds by playing nodes $t_0 < t_1 < \ldots < t_n < \ldots$ $(n < \omega)$ in T such that $\mathrm{ht}(t_n) \geq \alpha_n$.

(4) The persistency-rank of T is ω_2, where the rank $\rho(t)$ is defined as follows: $\rho(t) \geq \alpha$ if for all $\beta < \alpha$ and $\gamma < \omega_1$ there is an extension t' of t in T of height $\geq \gamma$ such that $\rho(t') \geq \beta$. The rank of T is the rank of the root of T. ($\rho(t) = \alpha$ if $\rho(t) \geq \alpha$ but $\rho(t) \not\geq \alpha+1$)

Proof of the equivalences. (4) $\to$ (3) The strategy of II in the persistency game is the following: Suppose she has played t_n and I has played α_n. II maintains the conditions $ht(t_n) \geq \alpha_n$ and $\rho(t_n) = \omega_2$.

Suppose now I plays $\alpha_{n+1} > \alpha_n$. There is $t_{n+1} > t_n$ such that $ht(t_{n+1}) \geq \beta$ and $\rho(t_{n+1}) = \omega_2$ by mere cardinality arguments.

<u>Proof of the equivalence.</u> (1) $\to$ (4) is trivial.

(3) $\to$ (2) We define f. Suppose $t \in T_p$. The branch $B = \{t' \in T_p : t' \le t\}$ goes thru T_p choosing a unique sequence of branches of F of lengths $\alpha_0, \alpha_1, \dots, \alpha_n$. The tail of B is on the branch of length α_n. We let I play $\alpha_0, \alpha_0+\alpha_1, \alpha_0+\alpha_1+\alpha_2, \dots, \alpha_0+\dots+\alpha_n$. Let the responses of I be $t_0 < \dots < t_n$. Now ht$(t_i) \ge \alpha_i$. ~~Let t^* be the unique predecessor of t_n in T of ht $\alpha_0+\dots+\alpha_{n-1}$.~~ Let $f(t)$ be the unique predecessor of t_n in T of ht ht(t). Suppose now $t' < t$ in T. The sequence $\alpha_0', \dots, \alpha_{m}'$ is an initial segment of $\alpha_0, \dots, \alpha_n$. Thus $f(t') < f(t)$.

$(2) \to (1)$ Let $T' = \{f(t) : t \in T_p\}$. If $f(t) \in T'$ and $\alpha < ht(T)$, there is an extension t' of t in T_p of ht $\geq \alpha$ and then $f(t')$ is an extension of $f(t)$ of ht $\geq \alpha$. $\square$

<u>Lemma</u> For all trees T exactly one of the following holds:

(1) $T_p \leq T$ (i.e. T is persistent)

(2) $T << T_p$, where $T << T'$ means

$\sigma T \leq T'$ and σT is the tree of

<u>Note:</u> Note that (1) and (2) cannot both hold, for o/w $T_p \leq T$ and $\sigma T \leq T_p$, whence $\sigma T \leq T$.

We call a tree T a <u>bottleneck</u> if for all trees T' either $T \le T'$ or $T' \ll T$. If T is in $\mathcal{T}_{\omega_1}$ and the above holds for T' in $\mathcal{T}_{\omega_1}$, we say that T is a bottleneck of $\mathcal{T}_{\omega_1}$. The above lemma shows that T_p is a bottleneck of $\mathcal{T}_{\omega_1}$.

<u>Question</u>: Are there bottlenecks in $\mathcal{T}_{\omega_1}$ above T_p?

<u>Theorem</u> [32,33] Suppose $\mathbb{P}$ adds $\aleph_2$ Cohen subsets to $\aleph_1$. Then $\mathbb{P} \Vdash$ "There are no bottlenecks in $\mathcal{T}_{\omega_1}$ above T_p".

<u>Proof</u> Suppose G is $\mathbb{P}$-generic and $T_1 \in V[G]$ is in $V[G]$ a bottleneck, $|T_1| = \aleph_1$. Let $\alpha < \omega_2$ so that $T_1 \in$ V in the canonical way

$V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$, $\mathbb{P}_\alpha$ consisting of the first α components of the product forcing $\mathbb{P}$. We can think of $V[G_{\alpha+1}]$ as $V[G_\alpha][A]$, where A is a Cohen-generic subset of ω_1, over $V[G_\alpha]$. We show that $V[G_{\alpha+1}] \models T(A) \not\leq T_1$. For this end let $\mathbb{Q}$ be $T(A)$ as a forcing notion and let H be $\mathbb{Q}$-generic over $V[G_{\alpha+1}]$. In $V[G_{\alpha+1}][H]$ the tree $T(A)$, and hence the tree T_1, has an uncountable branch. However, the product of Cohen forcing and $\mathbb{Q}$ has a σ-closed dense subset, so it cannot add a long branch to T_1. Thus neither can the Cohen forcing. In conclusion $T(A) \not\leq T_1$ in $V[G_{\alpha+1}]$, hence also in $V[G]$. Since T_1 is a bottleneck in $V[G]$, we have

$T_1 \leq T(A)$. Let us look at $T(-A)$. Since T_1 is a bottleneck, either $T(-A) \ll T_1$ or $T_1 \leq T(-A)$. The case $T(-A) \ll T_1$ (even $T(-A) \leq T_1$) is impossible. So $T_1 \leq T(-A)$. In conclusion $T_1 \leq T(A) \otimes T(-A)$

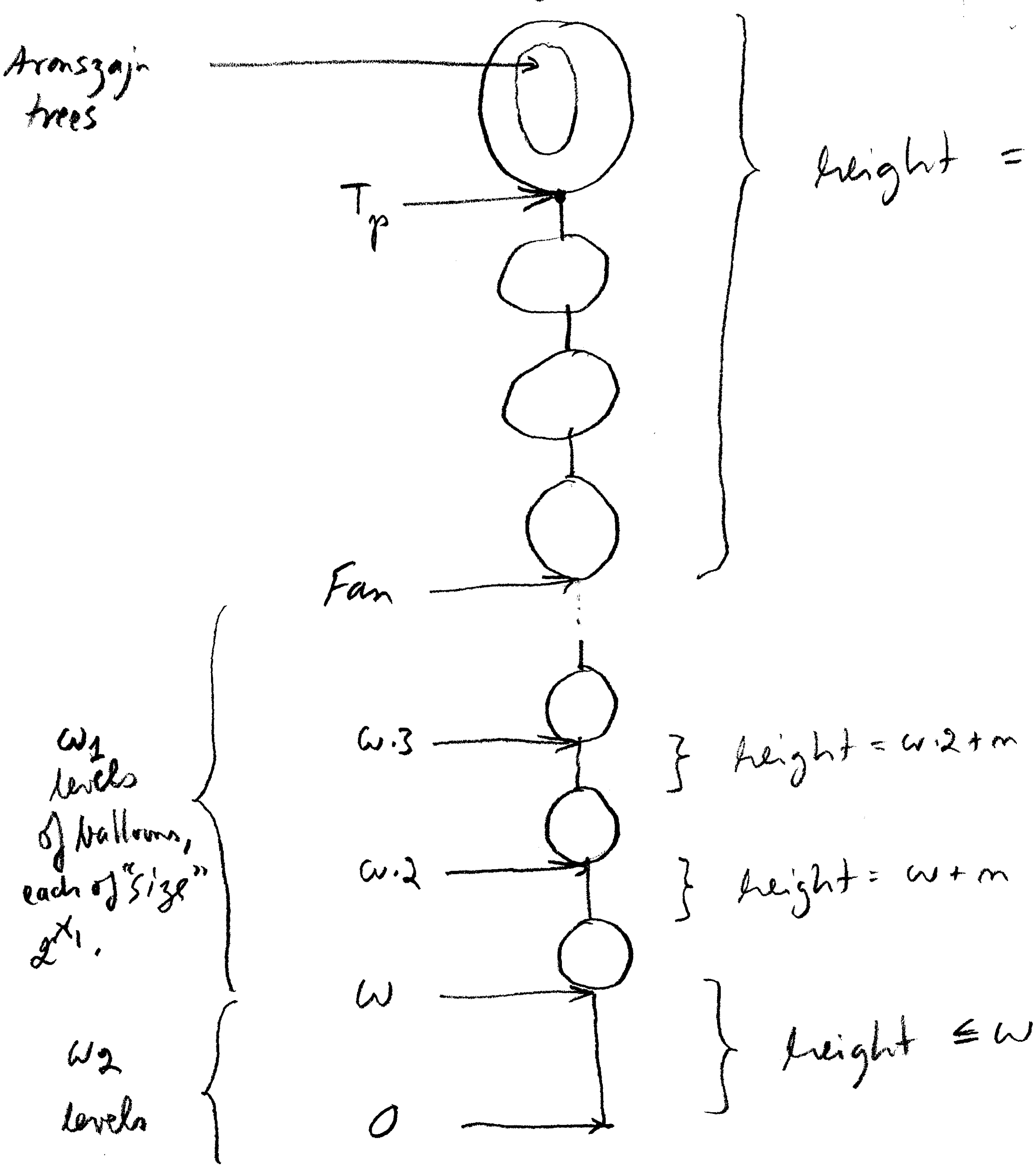
So to conclude we just need to prove

<u>Lemma</u> $T(A) \otimes T(-A) \leq T_\rho$ (A bistat.)

Proof We show that II wins the Comparison Game. Suppose I plays $(s_0, s_0') \in T(A) \otimes T(-A)$. Let $\delta_0 = \max(\max(s_0), \max(s_0'))$. Now II picks a branch of length δ_0 in T_ρ and keeps playing it for the next δ_0 moves. After these moves I has had to have played in both $T(A)$ and $T(-A)$ a closed sequence of length $\geq \delta_0$. After these δ_0

moves II switches to another similarly chosen branch. II can keep switching branches ω times. After this she has won the game as I cannot continue in $T(A) \otimes (-A)$ (since $A \cap -A = \emptyset$). $\square$

So the picture of T_1 is:

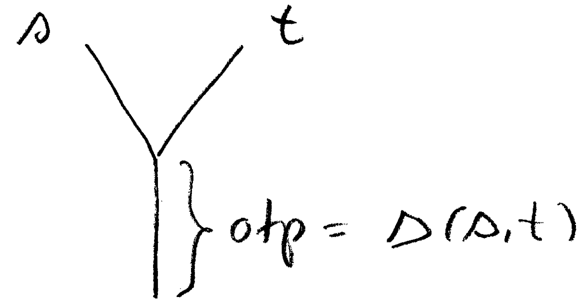


α^*

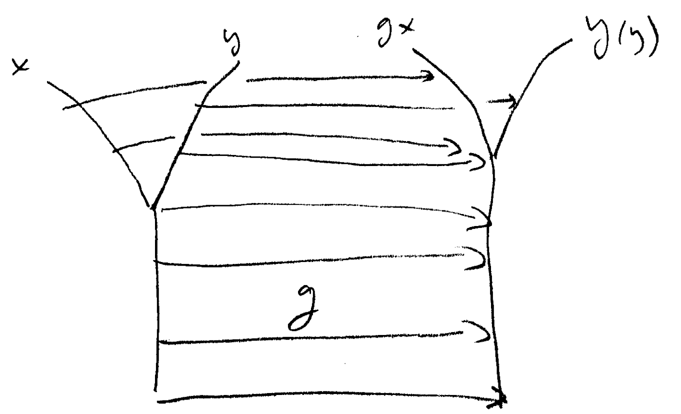
We get a better picture if we assume PFA as has been shown by Todorcevic [56]:

If T is a tree, define

$$\Delta(s,t) = \mathrm{otp}\,\{x \in T : x < s \text{ and } s < t\}$$



A partial map $g : S \to T$ is <u>Lipschitz</u> if for all x and y: $\Delta(x,y) \le \Delta(gx, gy)$



Note that if $g : S \to T$ is total and level & order-preserving, it is Lipschitz.

T is <u>Lipschitz</u> if for all level-preserving $g : S \to T$, $S \subseteq T$, $|S| \ge \aleph_1$ there is an