1.2. Uncountable models

Let us then move on to uncountable models. The natural space for the study of models of size $\aleph_1$ is $\omega_1^{\omega_1}$ with the basic open sets ([60], [32])

$$N(s,\alpha) = \{ f \in \omega_1^{\omega_1} : f \upharpoonright \alpha = s \}$$

where $s \in \omega_1^{\alpha}$. Note that $|\omega_1^{<\omega_1}| = 2^{\omega}$ while $|\omega^{<\omega}| = \omega$. $\aleph_1$-additive space, see page (?)

To see that this is a meaningful space let us first prove

Baire Category Theorem [5], [6]: If D_α, $\alpha < \omega_1$, are open dense, then $\bigcap_{\alpha<\omega_1} D_\alpha$ is dense in $\omega_1^{\omega_1}$. (So $\omega_1^{\omega_1}$ is not meager!) (More: Every comeager set has $2^{\aleph_1}$ els.)

Proof Let $f_0 \in D_0 \cap N(s,\alpha)$, $\alpha_0 < \omega_1$ so that $\alpha \le \alpha_0$ and suppose $N(s,\alpha)$ is given

$N(f_0\restriction\alpha_0, \alpha_0) \subseteq D_0$. Let $s_0 = f_0\restriction\alpha_0$. (15

If $f_\beta, \alpha_\beta, s_\beta$ have been chosen so that

$$s_0 \lhd s_1 \lhd \cdots \lhd s_\beta \lhd \cdots \quad \beta < \gamma$$
$$\alpha_0 < \alpha_1 < \cdots < \alpha_\beta < \cdots \quad \beta < \gamma$$
$$f_\beta\restriction\alpha_\beta = s_\beta$$
$$N(s_\beta, \alpha_\beta) \subseteq D_\beta$$

Let $f_\gamma \in D_\gamma$ s.t. $f_\gamma\restriction\alpha_\beta = s_\beta$ for all $\beta < \gamma$. Let $\alpha_\gamma < \omega_1$ so that $N(f_\gamma\restriction\alpha_\gamma, \alpha_\gamma) \subseteq D_\gamma$, and $s_\gamma = f_\gamma\restriction\alpha_\gamma$. Clearly there is f s.t. $f_\beta\restriction\alpha_\beta = s_\beta$ for all β and then $f \in N(s,\alpha) \cap \bigcap_\alpha D_\alpha$. $\square$

Let us call a set $A \subseteq \omega_1^{\omega_1}$ <u>co-analytic</u> if there is a tree $T \subseteq \omega_1^{<\omega_1} \times \omega_1^{<\omega_1}$ so that

$$f \in A \iff T(f) \text{ has an uncountable branch.}$$

Note that in general $|T(f)| \leq 2^{\omega}$.
Clearly, a set is co-analytic iff there is an open $B \subseteq \omega_1^{\omega_1} \times \omega_1^{\omega_1}$ s.t.

$$f \in A \iff \forall g\,((f,g) \in B).$$

Namely, if such a B exists, we can let

$$R = \{(s,s') : N((s,s'),\alpha) \subseteq B,\ \alpha < \omega_1,\ s,s' \in \omega_1^{\alpha}\}.$$

Then $$f \in A \iff \forall g\, \exists \alpha\, R(\bar{f}(\alpha), \bar{g}(\alpha)).$$

Let $$T = \{(s,s') : \forall \beta \leq \alpha\ \neg R(s \restriction \beta, s' \restriction \beta),\ \alpha < \omega_1,\ s,s' \in \omega_1^{\alpha}\}$$

Then

$f \in A \iff T(f)$ has an uncountable branch.

Conversely, we can construct an open B from T by first defining

$$R(s,s') \iff \forall \beta \leq \alpha \ (s \restriction \beta, s' \restriction \beta) \notin T$$

and then letting

$$B = \bigcup \{ N((s,s'),\alpha) : R(s,s'), \\ s,s' \in \omega_1^{\alpha}, \alpha < \omega_1 \}$$

So co-analytic sets in $\omega_1^{\omega_1}$ have a tree representation (by definition, but also definition in terms of projection).

Suppose now B is analytic ($\underset{\sim}{\Sigma}^1_1$) and S is a tree on $\omega_1^{<\omega_1}$ such that

$f \in B \iff S(f)$ has an uncountable branch.

Let

$$T' = \{ (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)) : \bar{g}(\alpha) \in T(f), \bar{h}(\alpha) \in S(f) \}.$$

If T' has an uncountable branch (f, g, h) then $f \in A \setminus B$, $\mathcal{K}$. So T' is a tree of size $\leq 2^{\omega}$ without uncountable branches. If $f \in B$, then there is an uncountable branch h in $S(f)$. Now

$$\bar{g}(\alpha) \mapsto (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$$

is an order-preserving mapping $T(f) \to T'$. We write $T(f) \leq T'$.

Let for any tree T_0 w/o uncountable branches

$$A_{T_0} = \{f \in A : T(f) \leq T_0\}.$$

Thus $A = \bigcup_{T_0} A_{T_0}$. CH implies A_{T_0} is $\underset{\sim}{\Sigma}^1_1$. Important to study universal families of trees!

Thus if $B \subseteq A$ is $\underset{\sim}{\Sigma}^1_1$, there is a tree T_0 (namely the above T') s.t. $B \subseteq A_{T_0}$. We have proved the Covering Theorem[32] for $\omega_1^{\omega_1}$. Moreover, if A itself is $\underset{\sim}{\Delta}^1_1$, then there is T_0 s.t. $A = A_{T_0}$ (Separation, too.). But are the sets A_T Borel in $\omega_1^{\omega_1}$? Let us define the Borel sets of $\omega_1^{\omega_1}$ as the smallest class containing the open sets and closed under

complement and union of length ω_1.
Borel sets are $\underset{\sim}{\Delta}^1_1$.

__Example__ $CUB = \{ f \in \omega_1^{\omega_1} : f(\alpha) = 0$ for a club of $\alpha \}$

$NS = \{ f \in \omega_1^{\omega_1} : f(\alpha) \neq 0$ for a club of $\alpha \}$

These are disjoint $\underset{\sim}{\Sigma}^1_1$ sets

__Theorem__ (Shelah-V. [45] 2000) Assume CH. Then CUB and NS cannot be separated by a Borel set. (But consistently CUB can be $\underset{\sim}{\Delta}^1_1$ (Mekler-Shelah [33])).

__Proof__ Every Borel set A has a "Borel code" c: $A = B_c$. Suppose $A = B_c$ separates CUB and NS ($CUB \subseteq A$, $A \cap NS = \emptyset$). Let $\mathbb{P}$ be Cohen forcing for adding a subset of ω_1. Let G be $\mathbb{P}$-generic. Let $g = \bigcup G$.

$V[G] \models \{\alpha : g(\alpha) = 0\}$ is leistat.

Either $\{\alpha : g(\alpha) = 0\}$ is in B_c or it is not in B_c. Assume w.l.o.g. that $\{\alpha : g(\alpha) = 0\}$ is in $A = B_c$. Let $p \in G$ such that

$$p \Vdash \{\alpha : \tilde{g}(\alpha) = 0\} \in B_{\check{c}}.$$

Let μ be large and $M \prec (H(\mu), \in, <^*)$ where $<^*$ is a well-order of $H(\mu)$. We assume $\omega_1, p, \mathbb{P}, T((c) \in M$, ${}^{\omega}M \subseteq M$ (here we use CH) and $|M| = \omega_1$.

Now it is easy to construct a generic G' over M in V such that $p \in G'$ and $\{\alpha : \tilde{g}^{G'}(\alpha) = 0\} \in NS$. It is easy to show that $B_c = (B_{\check{c}})^{G'}$.

Since $M \models$ "$p \Vdash \tilde{g}(\alpha) = 0 \in B_{\check{c}}$",

$\{\alpha : \tilde{g}^{G'}(\alpha) = 0\} \in B_c$. This contradicts $B_c \cap NS = \emptyset$. $\square$

<u>Theorem</u> (Shelah-V. 2000 [45]) $MA + \neg CH$ implies CUB is definable in $L_{\omega_1\omega_1}$.

<u>Note</u>: $L_{\omega_2\omega_1} \subseteq$ Borel if we assume CH.

<u>Theorem</u> (Halko-Shelah [6]) CUB is not Borel. (But it can be $\underset{\sim}{\Delta}^1_1$ [33])

As in the case of countable models, the relation $\mathfrak{A} \cong \mathfrak{B}$ is an analytic equivalence relation in ${\omega_1}^{\omega_1}$, and the orbits

$$I(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{B} \cong \mathfrak{A}\}$$

are also analytic. We can construct the tree T of attempts

$$(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)) \qquad (*)$$

to build an isomorphism f between models g and h of size $\aleph_1$. Let

Coded by

$$\mathcal{B} \not\equiv_{T_0} \mathcal{A} \iff T(\mathcal{B},\mathcal{A}) \leq T_0$$

The relation $\mathcal{B} \not\equiv_{T_0} \mathcal{A}$ has a back-and forth characterization, like $\mathcal{B} \not\equiv_\alpha \mathcal{A}$. Due to some mathematical facts discussed in Lectures II & III, $\mathcal{B} \not\equiv_{T_0} \mathcal{A}$ is not the negation of $\mathcal{B} \equiv_{T_0} \mathcal{A}$, which can be defined as $\sigma T_0 \leq T(\mathcal{B},\mathcal{A})$[1]. See later for a definition of σT_0.

The existence of an analogue of Scott ranks of models becomes dependent on the stability theoretic properties of the model.

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1) $T(\mathcal{B},\mathcal{A})$ is the tree T of $(*)$ when g codes $\mathcal{B}$ and h codes $\mathcal{A}$.

A *different* approach would be to study the space $\mathbb{R}^{<\omega_1}$. Steel has proved that if enough large cardinals exist and $T \subseteq \mathbb{R}^{<\omega_1}$ is in $L(\mathbb{R})$, then "T has an uncountable branch" is forcing absolute. It follows that if $\mathfrak{A}$ and $\mathfrak{B}$ are models in $L(\mathbb{R})$ and their universe is ω_1, then "$\mathfrak{A} \cong \mathfrak{B}$" is forcing absolute. Thus, if we can force them isomorphic w/o collapsing $\aleph_1$, they are isomorphic.

To avoid dependence on CH we can consider the space λ^{ω_1} where λ is a s.s.l. of cofinality ω_1. Then $|\lambda^{\omega}| = \lambda$ and $|2^{\lambda}| = |\lambda^{\omega_1}|$. Thus trees $T \subseteq \lambda^{<\omega_1}$ can be identified with elements of λ^{ω_1}. The basic open sets are again

$$N(s, \alpha) = \{f \in \lambda^{\omega_1} : f\restriction\alpha = s\}$$

where $s \in \lambda^{<\omega_1}$. The Covering Theorem holds again. In particular, every $\underset{\sim}{\Pi}^1_1$ set A can be covered by sets A_T, T a tree of size λ with no uncountable branches, such that A is $\underset{\sim}{\Delta}^1_1$ iff $A = A_T$ for some such T.

Shelah [Sh #80] (1978) introduced a Generalized Martin's Axiom for ω_1, GMA, and proved: If GMA holds, then the meager ideal on 2^{ω_1} is closed under unions of length $< 2^{\aleph_1}$.

Halko–Shelah [6]: CUB does not have the Baire property (although it is $\underset{\sim}{\Sigma}^1_1$). Borel sets have the Baire property, so CUB is not Borel.

What else is known about $\omega_1^{\omega_1}$, or κ^λ?

$\mathcal{T}_\kappa$ = trees of size and height κ

$U(\kappa)$ (Universality property) there is a family $\mathcal{U} \subseteq \mathcal{T}_{\omega_1}$ s.t. $|\mathcal{U}| = \kappa$ and $\forall T \in \mathcal{T}_{\omega_1}$ $\exists T' \in \mathcal{U}\,(T \leq T')$. $U(2^{\aleph_1})$. $CH \Rightarrow \neg U(\aleph_1)$

$B(\kappa)$ (Bounding property) every family $\mathcal{B} \subseteq \mathcal{T}_{\omega_1}$ s.t. $|\mathcal{B}| = \kappa$ is bounded i.e. $\exists T \in \mathcal{T}_{\omega_1} \forall T' \in \mathcal{B}\,(T' \leq T)$. $B(\aleph_2)$

$(CH \wedge U(\kappa) \wedge B(\lambda)) \Rightarrow \kappa \geq \lambda$

$CP(\kappa)$ (Covering property) If $A \subseteq \omega_1^\omega$ is $\underset{\sim}{\Pi}^1_1$, then there are $\underset{\sim}{\Sigma}^1_1$ sets A_α, $\alpha < \kappa$, such that $A = \bigcup_\alpha A_\alpha$ and if $B \subseteq A$ is $\underset{\sim}{\Sigma}^1_1$, then $B \subseteq A_\alpha$ for some $\alpha < \kappa$.

$CH + U(\kappa) \Rightarrow CP(\kappa)$

$CH + B(\kappa) \Rightarrow \forall \lambda < \kappa\, \neg CP(\lambda)$.

Mekler–V. 1993: $U(\kappa) \wedge B(\kappa)$ is consistent for any given κ, $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$.