Set theoretic model theory

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Part 0. Introductory words for the tutorial

When we want to express mathematical concepts using logic we can do several things:

(1) Increase the vocabulary and add new axioms.
- study models: Model theory
- study proofs: Proof theory
- ZFC: Set theory

(2) Define a concept of algorithm appropriate for the task.
- study computability: Recursion theory
- study complexity: Complexity theory

(3) Extend the concept of model
- Modal logic
- Banach space model theory
- Topological model theory

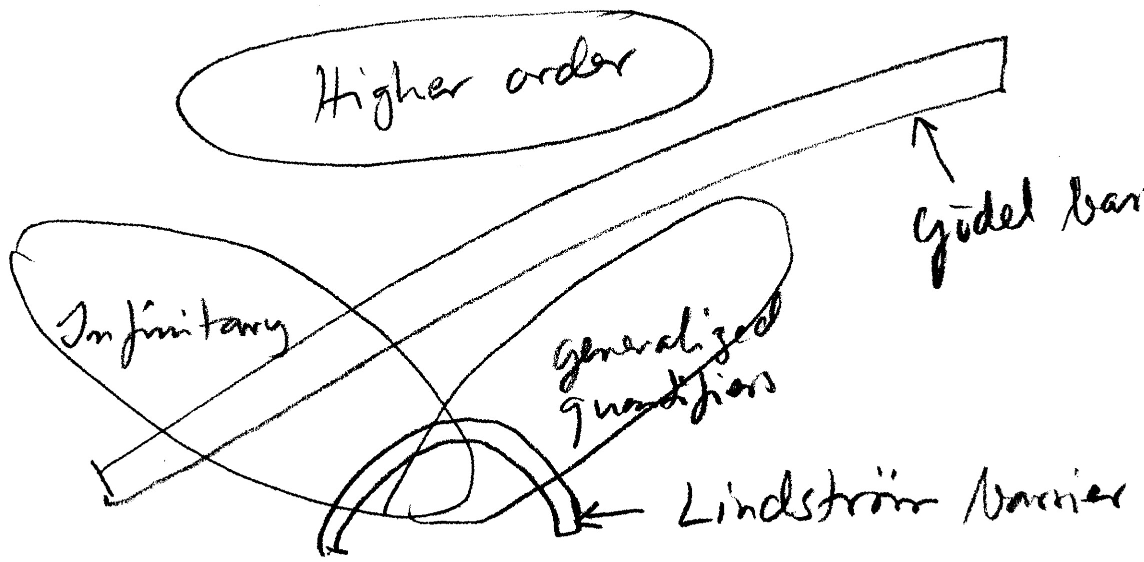
(4) Add new logical operations

- Infinitary logic
- Generalized quantifiers
- Higher order logic

Example
1) Uncountability
   - ZFC, Gödel completeness
   - $\mathcal{L}Q_1$, Keisler completeness
2) Infinity
   - ZFC
   - $\mathcal{L}Q_0$
   - $\mathcal{L}_{\omega_1\omega}$

The approach of this tutorial is (4).
This is sometimes called abstract logic.

The landscape of logics



The Gödel barrier is a bit like the Main Gap of Shelah in model theory.

Our study of the topology of the space of models is based on the primitive idea that models that ~~are~~ can be separated by topologically simple (e.g. Borel) sets are also logically different, different in their logical properties. We do not study one single model, but sets of models (e.g. orbits).

Part 1 The topology of uncountable models

Recap: Countable models

11. In this lecture we review some basic facts about the topology of countable models and introduce machinery to develop topology of uncountable models. It turns out that (when we move to the uncountable) the concept of a tree without cofinal branches turns out to play a decisive role. Why uncountable structures? Non-structure!

The classical Baire space is the set ω^ω with the basic neighborhoods

$$N(s,n) = \{ f : f\restriction n = s \}$$

where $s \in \omega^{<\omega}$. We use the notation

$$\bar{f}(n) = \langle f(0), \ldots, f(n-1) \rangle.$$

We consider trees on $\omega^{<\omega}$ (and $\omega^{<\omega} \times \omega^{<\omega}$), i.e. downward closed sets of sequences (or pairs of sequences). If $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ is a tree and $f \in \omega^\omega$, then

$$T(f) = \{ \bar{g}(n) : (\bar{f}(n), \bar{g}(n)) \in T \}.$$

A set $A \subseteq \omega^\omega$ is <u>analytic</u> iff there is a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that

$$f \in A \iff T(f) \text{ has no infinite branch}$$

This is the <u>tree-representation</u> of co-analytic sets. Let

$$A_\alpha = \{ f \in A : \mathrm{rank}(T(f)) \leq \alpha \}.$$

So $A = \bigcup_{\alpha < \omega_1} A_\alpha$ (note that $\mathrm{card}(T) \leq \aleph_0$).

If $B \subseteq A$ is Σ^1_1, and S' is a tree such that

$$f \in B \iff S(f) \text{ has an infinite branch}$$

then let

$$T' = \{ (\bar{f}(n), \bar{g}(n), \bar{h}(n)) : \bar{g}(n) \in T(f),\ \bar{h}(n) \in S(f) \}.$$

If T' has an infinite branch (f, g, h), then $f \in A \setminus B$, ↯. So T' has some rank $\alpha < \omega_1$. If $f \in B$, then there is an infinite branch h in $S(f)$. If $\bar{g}(n) \in T(f)$, then $(\bar{f}(n), \bar{g}(n), \bar{h}(n)) \in T'$ and the mapping

$$\bar{g}(n) \mapsto (\bar{f}(n), \bar{g}(n), \bar{h}(n))$$

is an order-preserving mapping $T(f) \to T'$.

So $f \in A_\alpha$. This is the <u>Covering Theorem</u> i.e. any $\underset{\sim}{\Sigma}^1_1$ subset of A is covered by one of the A_α, $\alpha < \omega_1$. In particular, A itself is $\underset{\sim}{\Delta}^1_1$ iff there is $\alpha < \omega_1$ such that $A = A_\alpha$. Since each A_α is Borel, this gives the <u>Souslin-Kleene Theorem</u>: $\underset{\sim}{\Delta}^1_1 =$ Borel. Also <u>Separation Theorem</u> for $\underset{\sim}{\Sigma}^1_1$.

We note w/o proof the theorem of <u>Silver</u> that every $\underset{\sim}{\Pi}^1_1$ equivalence relation E satisfies $E \leq_B \mathbb{N}$ or $\mathbb{R} \leq_B E$. The <u>Generalized Glimm-Effros Dichotomy</u> says that if E is a Borel equivalence relation then $E \leq_B \mathbb{R}$ or $E_0 \leq_B E$, where $x E_0 y \Leftrightarrow x-y \in \mathbb{Q}$.

A good example of an analytic equivalence relation is the isomorphism relation between countable ~~structures~~ models of a theory on $L_{\omega_1\omega}$.

If $A = \{(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \not\cong \mathfrak{B}, \mathfrak{A}, \mathfrak{B} \models T\}$ A_α is as above, then A_α corresponds to non-isomorphism among models of T of <u>Scott-rank</u> $\leq \alpha$. Many Borel equivalence relations E are Borel-reducible to isomorphism of countable structures. We say then that E is classifiable by countable models. There are Borel equivalence relations that are <u>not</u> classifiable by countable models. Hjorth has developed a theory of <u>turbulence</u> which shows that conjugacy on the group of homeomorphisms of $[0,1]$ is classifiable by countable models but of $[0,1]\times[0,1]$ is not.

Let T be the tree of attempts

$$(\bar{f}(n), \bar{g}(n), \bar{h}(n))$$

to build an isomorphism f between countable models (coded by) g and h.

Thus

$$\mathcal{B} \not\cong \mathcal{O} \iff T(\mathcal{B}, \mathcal{O}) \text{ has no infinite branch}$$

The least α in the Scott-Watershed.

Let $\mathcal{B} \equiv_\alpha \mathfrak{A} \iff \mathrm{rank}(T(\mathcal{B}, \mathfrak{A})) \leq \alpha$.

The relation $\mathfrak{A} \equiv_\alpha \mathcal{B}$ is the usual elementary equivalence of $\mathfrak{A}$ and $\mathcal{B}$ in $L_{\omega_1\omega}$ up to quantifier rank α.

The Scott-rank of $\mathfrak{A}$ is the least α s.t.

$$(\mathfrak{A}, \vec{a}) \equiv_\alpha (\mathfrak{A}, \vec{b})$$

whenever $(\mathfrak{A}, \vec{a}) \not\equiv (\mathfrak{A}, \vec{b})$.

If $\mathfrak{A} \equiv_{\alpha+\omega} \mathcal{B}$ for this α, then $\mathfrak{A} \cong \mathcal{B}$ as a back-and-forth argument shows. This is Scott's Isomorphism Theorem. The Scott ranks put countable models into a hierarchy of length ω_1. The scott-rank is an important invariant (ordinal-) of a countable model.

Following a back-and-forth characterization of $\mathfrak{A} \equiv_\alpha \mathfrak{B}$, one can easily write for each countable model $\mathfrak{A}$ a <u>Scott sentence</u> $\theta_{\mathfrak{A}}$ s.t.

$$\mathfrak{B} \models \theta_{\mathfrak{A}} \iff \mathfrak{B} \cong \mathfrak{A}.$$

It is crucial here that we can replace $\mathfrak{B} \cong \mathfrak{A}$ by $\mathfrak{B} \equiv_\alpha \mathfrak{A}$ where α is the Scott rank of $\mathfrak{A}$.

Scott observed that every class of models defined by a sentence of $L_{\omega_1\omega}$ is Borel. Lopez-Escobar proved Interpolation Theorem for $L_{\omega_1\omega}$:

If $\varphi(P,Q) \models \psi(P,R)$, then there is $\theta(P)$ such that $\varphi(P,Q) \models \theta(P)$ and $\theta(P) \models \psi(P,R)$.

It follows from the interpolation theorem that invariant Borel sets are exactly the $L_{\omega_1\omega}$-definable sets of models. Hence $\underset{\sim}{\Sigma}^1_1$-sets (invariant) correspond to $PC(L_{\omega_1\omega})$-definable sets and the Interpolation theorem becomes the Separation Property for $\underset{\sim}{\Sigma}^1_1$. Vaught extended this to a Covering Theorem for $L_{\omega_1\omega}$ and more generally, descriptive set theory of $L_{\omega_1\omega}$.

<u>Note</u> The <u>orbit</u> of a countable model $\mathfrak{A}$ is

$$I(\mathfrak{A}) = \{B : \mathfrak{A} \cong B\}.$$

This is always Borel by the Scott Isomorphism Theorem: $I(\mathfrak{A}) = \{B : B \models \theta_{\mathfrak{A}}\}$.