Research Statement for Young Set Theory Meeting

John Krueger

My research in set theory is focussed on consistency results, forcing, and combinatorics. I am especially interested in combinatorial properties related to stationary sets, for example, saturated ideals, the approachability property, and internally approachable models. In forcing I specialize in proper forcing style methods, iterated forcing, and methods for extending elementary embedding. I give two examples of open problems I am interested in.

The first problem concerns consistency results for saturated ideals. A famous theorem of set theory is the consistency of the statement that the non-stationary ideal on ω_1 is saturated. This statement means that there does not exist a collection of stationary subsets of ω_1 with size $\aleph_2$, which is an *antichain* in the sense that the intersection of any two sets in the collection is non-stationary. A natural problem is to generalize this consistency result to cardinals larger than ω_1. At a first glance, this appears to be impossible. For example, Shelah has proven that the non-stationary on ω_2 cannot be saturated, because any stationary subset of the set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ can be split into $\aleph_3$ many stationary subsets, any two of which have non-stationary intersection. However, it may be that the statement "the non-stationary ideal on ω_2 is saturated" is the wrong generalization of the saturation of the non-stationary ideal on ω_1. Since the limit ordinals below ω_1 all have cofinality ω, the non-stationary ideal on ω_1 is the *same ideal* as the non-stationary ideal on ω_1 restricted to ordinals with cofinality ω. Perhaps then the correct generalization is the statement that the non-stationary ideal on ω_2 restricted to cofinality ω_1 is saturated. Whether this statement about ω_2 is consistent is a well-known open problem in set theory.

The second problem I discuss involves singular cardinal combinatorics and forcing. Define the *approachability ideal* $I[\aleph_{\omega+1}]$ as the collection of sets $A \subseteq \aleph_{\omega+1}$ such that there exists a sequence $\langle a_i : i < \aleph_{\omega+1} \rangle$ of bounded subsets of $\aleph_{\omega+1}$ such that for club many α in A, there is an unbounded set $c \subseteq \alpha$ with order type equal to $\mathrm{cf}(\alpha)$ such that every initial segment of c is equal to a_i for some $i < \alpha$. An important theorem in singular cardinal combinatorics is Shelah's result that for all $n < \omega$, $I[\aleph_{\omega+1}]$ contains a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_n)$. If $\Box_{\aleph_\omega}$ holds, then every subset of $\aleph_{\omega+1}$ is in $I[\aleph_{\omega+1}]$. On the other hand, Magidor proved that under Martin's Maximum, there is a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_1)$ which is not in $I[\aleph_{\omega+1}]$. A current open problem is whether $\aleph_{\omega+1} \cap \mathrm{cof}(> \omega_1)$ is in $I[\aleph_{\omega+1}]$. Personally I believe this is most likely false in general. But constructing a model in which, for example, there is a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_2)$ which is not in $I[\aleph_{\omega+1}]$ turns out to be quite difficult.