Forcing absoluteness and Regularity properties

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1. Backgrounds and results.

Forcing absoluteness is one of the central topics in set theory and it connects many areas in set theory while regularity properties are nice properties for sets of reals which have been deeply investigated for many years. There is a close connection between forcing absoluteness and regularity properties, e.g. all the Σ^1_3-formulas are absolute between V and its Cohen forcing extensions iff every $\boldsymbol{\Delta}^1_2$-set of reals has the Baire property iff for any real a, there is a Cohen real over $L[a]$. The same kind of equivalence holds for random forcing and Lebesgue measurability, Hechler forcing and the Baire property for dominating topology, and Mathias forcing and Ramsey property etc. In my master's thesis [3], I proved the equivalence for Sacks forcing and Bernstein property (a set of reals has the Bernstein property if it or the complement of it contains a perfect set). In this case, the statement for generic reals cannot be added to the equivalence, i.e. the statement "for any real a, there is a Sacks real over $L[a]$" is stronger than the other two statements. Instead, the statement "for any real a there is a real which is not in $L[a]$" is equivalent to them. Recently, I have succeeded to introduce a large class of forcing notions from each of which we can define the corresponding regularity property and to prove the equivalence in each case in a uniform way. This class contains all the practical forcing notions and this result implies the unknown equivalence for some forcings (e.g. Miller forcing and Silver forcing). Also this result solves one open question in the paper "Silver measurability and its relation to other regularity properties" by Brendle, Halbeisen and Löwe [2].

2. Future directions and questions

(a) Generic reals.

Although the existence of generic reals for L is too strong for the equivalence in the case of Sacks forcing as I mentioned above, we have a weaker notion of generic reals so called "quasi-generic reals" and for these reals, the equivalence for the three statements holds even in the case of Sacks forcing. The question is if we can generalize this relationship of three statements up to any forcing in the class I have introduced. The answer is true if we restrict our attention to ccc $\boldsymbol{\Sigma}^1_2$ forcings.

(b) Higher level forcing absoluteness and regularity properties.

The relation I have mentioned is only for Σ^1_3-forcing-absoluteness and regularity properties for $\boldsymbol{\Delta}^1_2$ sets of reals. The question is if we can generalize this relationship up to Σ^1_{n+1}-forcing-absoluteness and regularity properties for $\boldsymbol{\Delta}^1_n$ sets of reals. When $n = 3$, we have affirmative answers for Cohen forcing, random forcing and Sacks forcing although we need large cardinal assumptions. The goal of this question is to find the optimal assumption to prove the equivalence for each n.

References

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[2] Jörg Brendle, Lorenz Halbeisen, Benedikt Löwe, Silver measurability and its relation to other regularity properties, Mathematical Proceedings of the Cambridge Philosophical Society 138 (2005), p. 135-149

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http://www.illc.uva.nl/Publications/ResearchReports/PP-2006-12.text.pdf

[4] Daisuke Ikegami, Forcing absoluteness and regularity properties, in preparation