Research statement

Models of topological set theory

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In Zermelo-Fraenkel set theory the contradictory naive comprehension scheme is weakened to the separation scheme, that is, it is restricted to formulas of the form $x \in a \wedge \varphi(x)$. In positive set theories one restricts the comprehension scheme to another class of formulas, namely the generalized positive ones, which are defined recursively just like formulas in general, but with the negation step weakened to bounded universal quantification. Since intersections, binary unions and the universe $\mathbb{V}$ are all defined by generalized positive formulas, this axiom scheme implies (given that the empty set exists) that the universe is a topology on itself. Moreover, it turns out to be finer than or equal to its own exponential topology. Conversely, the topological set theory **TopS** consists of topological axioms from which the generalized positive comprehension scheme follows. Its objects are classes, for which extensionality, full comprehension and a global choice principle hold. For the class of all sets $\mathbb{V}$, the following purely topological axioms are postulated:

- $\mathbb{V}$ is a κ-compact κ-topology on itself in the sense of closed sets, where κ=On.
- $\mathbb{V}$ is its own exponential topology.

Since in a κ-compact κ-topology the sets of size less than κ are exactly the discrete ones, the first axiom can be formulated in terms of discreteness without referring to the concept of an ordinal number. As an axiom of infinity, one can demand that the class of natural numbers ω be a set.

In my diploma thesis I showed how the usual set theoretic constructions can be carried out in **TopS**, proved some regularity properties of the universe's topology and roughly estimated its consistency strength by giving two natural models in **ZFC** with a weakly-compact cardinal, and in **TopS** a model of Kelley-Morse set theory in which the class of ordinals is weakly-compact.

I now intend to further determine its consistency strength and hopefully even find a variant of Kelley-Morse set theory which is mutually interpretable with **TopS**. I also want to investigate variants of the two models or new models to find out about the dependencies of some additional axioms like:

- The universe's topology is induced by a natural ultrametric.
- There exists a (class-)well-order on the universe.
- The well-founded sets are dense in $\mathbb{V}$. (Foundation axiom)
- To every [finite/discrete] extensional structure $\langle A, e \rangle$ there exists a transitive set X such that $\langle X, \in \rangle$ is isomorphic to $\langle A, e \rangle$. (Antifoundation axiom)

Furthermore, in the context of positive set theories and exponential topologies it is natural to weaken the topological axioms in such a way that they don't imply the existence of the empty set. Such a theory might – unlike **TopS** – have models without isolated points. One might also admit urelements and see whether these weaker topological set theories have a lower consistency strength.