Models of Set Theory I. - Summer 2019

Problem 45 (8 Points)**.** Let M be a countable transitive model of ZFC, let κ be an uncountable regular cardinal in M and let $\mathbb{P} \in M$ be a separative partial order (see Problem 40). Show that, if $\mathbb{P}$ is not $<\kappa$-distributive in M, then there is G $\mathbb{P}$-generic over M and $f : \lambda \longrightarrow \mathrm{Ord}$ with $\lambda < \kappa$ and $f \in M[G] \setminus M$. (Hint: Work in M and fix a sequence $\langle D_\alpha \mid \alpha < \lambda \rangle$ of dense open subsets of $\mathbb{P}$ such that $\lambda < \kappa$ and $\bigcap_{\alpha<\lambda} D_\alpha$ is not dense in $\mathbb{P}$. Given $\alpha < \lambda$, let $\langle a^\alpha_\beta \mid \beta < \nu_\alpha \rangle$ enumerate a maximal antichain in D_α. Define

$$\sigma \ = \ \{(\mathsf{op}(\check{\alpha}, \check{\beta}), a^\alpha_\beta) \mid \alpha < \lambda, \ \beta < \nu_\alpha\} \in M^{\mathbb{P}}$$

and find G $\mathbb{P}$-generic over M such that $\sigma^G : \lambda \longrightarrow \mathrm{Ord}$ with $\sigma^G \notin M$).

Problem 46 (*Almost disjoint coding forcing*, 8 Points)**.** Given a subset A of the set ${}^\omega\omega$ of all functions from ω to ω, we define $\mathbb{P}_A$ to be the partial order whose conditions are pairs $p = (c_p, X_p)$ with

- X_p is a finite subset of A.
- $c_p : {}^{<\omega}\omega \xrightarrow{part} 2$ is a finite partial function.

such that $p \leq_{\mathbb{P}_A} q$ holds if and only if the following statements hold.

(1) $c_q \subseteq c_p$ and $X_q \subseteq X_p$.

(2) If $x \in X_q$ and $l < \omega$ with $x \restriction l \in \mathrm{dom}(c_p) \setminus \mathrm{dom}(c_q)$, then $c_p(x \restriction l) = 1$.

Prove the following statements:

(a) Given $x \in A$ and $t \in {}^{<\omega}\omega$, the set

$$\{p \in \mathbb{P}_A \mid x \in X_p \ \wedge \ t \in \mathrm{dom}(c_p)\}$$

is dense in $\mathbb{P}_A$.

(b) $\mathbb{P}_A$ satisfies the countable chain condition.

(c) Given $y \in {}^\omega\omega \setminus A$ and $k < \omega$, the set

$$\{p \in \mathbb{P}_A \mid \exists l < \omega \ [k < l \ \wedge \ y \restriction l \in \mathrm{dom}(c_p) \ \wedge \ c_p(y \restriction l) = 0]\}$$

is dense in $\mathbb{P}_A$.

Let M be a transitive model of ZFC, let $A \in \mathcal{P}({}^\omega\omega) \cap M$ and let G be $\mathbb{P}_A^M$-generic over M.

(d) There is a function $C : {}^{<\omega}\omega \longrightarrow 2$ in $M[G]$ such that the equivalence

$$x \in A \ \longleftrightarrow \exists N < \omega \ \forall n < \omega \ [N < n \ \longrightarrow \ C(x \restriction n) = 1]$$

holds for every $x \in \mathcal{P}({}^\omega\omega) \cap M$.

Problem 47 (6 Points)**.** Let $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ be a dense embedding of partial orders. By recursion on the strongly well-founded relation "$a \in \mathrm{tc}(b)$", we define a class function

$$\pi_* : \mathrm{V}^{\mathbb{P}} \longrightarrow \mathrm{V}^{\mathbb{Q}}; \tau \mapsto \{(\pi_*(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}.$$

Prove that, if $\varphi(v_0, \dots, v_{n-1})$ is an $\mathcal{L}_\in$-formula, then

$$p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \dots, \tau_{n-1}) \iff \pi(p) \Vdash^*_{\mathbb{Q}} \varphi(\pi_*(\tau_0), \dots, \pi_*(\tau_{n-1}))$$

holds for all $p \in \mathbb{P}$ and $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$.

Problem 48 (16 Bonus Points)**.** Let $\mathbb{B} = \langle \mathbb{B}, \leq, \wedge, \vee, 0, 1, ' \rangle$ be a complete boolean algebra, let $\mathbb{B}^*$ denote the corresponding partial order and let $\mathcal{U}$ be an ultrafilter on $\mathbb{B}$ (i.e. there is an homomorphism $\pi_{\mathcal{U}}$ of boolean algebras from $\mathbb{B}$ to the unique boolean algebra $\{0, 1\}$ with two elements such that $\mathcal{U} = \pi^{-1}``\{1\}$).
Define a relation $\equiv_{\mathcal{U}}$ on $\mathrm{V}^{\mathbb{B}^*}$ by setting

$$\sigma \equiv_{\mathcal{U}} \tau \iff [\![\text{"}\sigma = \tau\text{"}]\!]_{\mathbb{B}} \in \mathcal{U}$$

for all $\sigma, \tau \in \mathrm{V}^{\mathbb{B}^*}$.

(1) Show that $\equiv_{\mathcal{U}}$ is an equivalence relation on $\mathrm{V}^{\mathbb{B}^*}$.

Given $\tau \in \mathrm{V}^{\mathbb{B}^*}$, define

$$[\tau]_{\mathcal{U}} = \{\sigma \in \mathrm{V}^{\mathbb{B}^*} \mid \sigma \equiv_{\mathcal{U}} \tau \ \wedge \ \forall \rho \in \mathrm{V}^{\mathbb{B}^*} \ [\rho \equiv_{\mathcal{U}} \tau \longrightarrow \mathrm{rank}(\rho) \geq \mathrm{rank}(\sigma)]\} \in \mathrm{V}.$$

Let $\mathrm{V}/\,\mathcal{U}$ denote the class $\{[\tau]_{\mathcal{U}} \mid \tau \in \mathrm{V}^{\mathbb{B}^*}\}$ and define a relation $\in_{\mathcal{U}}$ on $\mathrm{V}/\,\mathcal{U}$ by setting

$$[\sigma]_{\mathcal{U}} \in_{\mathcal{U}} [\tau]_{\mathcal{U}} \Leftrightarrow [\![\text{"}\sigma \in \tau\text{"}]\!]_{\mathbb{B}} \in \mathcal{U}$$

for all $\sigma, \tau \in \mathrm{V}^{\mathbb{B}^*}$.

(2) Show that the relation $\in_{\mathcal{U}}$ is well-defined.

(3) Show that

$$(\mathrm{V}/\,\mathcal{U}, \in_{\mathcal{U}}) \models \varphi([\tau_0]_{\mathcal{U}}, \dots, [\tau_{n-1}]_{\mathcal{U}}) \iff [\![\varphi(\tau_0, \dots, \tau_{n-1})]\!]_{\mathbb{B}} \in \mathcal{U}$$

holds for every $\mathcal{L}_\in$-formula $\varphi(v_0, \dots, v_{n-1})$ and all $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$.

(4) Show that $(\mathrm{V}/\,\mathcal{U}, \in_{\mathcal{U}})$ is a model of ZFC.

Please hand in your solutions on Monday, July 01, before the lecture.