Problem 41 (6 Points)**.** Let M be a transitive model of ZFC, let κ be an uncountable regular cardinal in M, let $\mathbb{P} \in M$ be a partial order that satisfies the κ-chain condition in M and let G be $\mathbb{P}$-generic over M. Show that every stationary subset of κ in M is stationary in $M[G]$ (Hint: Pick $\sigma, \tau \in M^{\mathbb{P}}$ with the property that σ^G is a closed unbounded subset of κ in $M[G]$ and $\tau^G : \kappa \longrightarrow \kappa$ is the monotone enumeration of σ^G in $M[G]$. Use the name σ and the κ-chain condition to find a closed unbounded subset C of κ in M and a condition $p \in G$ with $(p \Vdash^*_{\mathbb{P}}$ "$\check{C} \subseteq \sigma$"$)^M$).

Problem 42. Given a non-principal ultrafilter $\mathcal{U}$ on ω, we define $\mathbb{P}_{\mathcal{U}}$ to be the partial order whose conditions are pairs $p = (s_p, A_p)$ such that $s_p : n_p \longrightarrow \omega$ is a strictly increasing function with $n_p < \omega$ and $A_p \in \mathcal{U}$ and whose ordering is given by

$$p \leq_{\mathbb{P}_{\mathcal{U}}} q \iff s_q \subseteq s_p \ \wedge \ A_p \subseteq A_q \ \wedge \ \forall k \in \operatorname{dom}(s_p) \setminus \operatorname{dom}(s_q) \ s_p(k) \in A_q.$$

(1) (3 Points) Show that $\mathbb{P}_{\mathcal{U}}$ satisfies the countable chain condition.

Let M be a transitive model of ZFC, let $\mathcal{U}$ be a non-principal ultrafilter on ω in M and let G be $\mathbb{P}^M_{\mathcal{U}}$-generic over M.

(2) (3 Points) Show that $s_G = \bigcup\{s_p \mid p \in G\}$ is a strictly increasing function with domain ω.

(3) (4 Points) Prove that

$$\mathcal{U} \ = \ \{A \in \mathcal{P}(\omega) \cap M \mid \operatorname{ran}(s_G) \setminus A \textit{ is finite}\}.$$

Problem 43. Let $\varphi(v_0, \ldots, v_n)$ be an $\mathcal{L}_{\in}$-formula, let $\mathbb{P}$ be a partial order and let p be a condition in $\mathbb{P}$. Prove the following statements:

(1) (2 Points) If $\sigma_0, \ldots, \sigma_n, \tau_0, \ldots, \tau_n \in \mathrm{V}^{\mathbb{P}}$ with

$$p \Vdash^*_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_n) \ \wedge \ \sigma_0 = \tau_0 \ \wedge \ \ldots \ \wedge \ \sigma_n = \tau_n,$$

then $p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_n)$.

(2) (*Maximality Principle*, 4 Points) If $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$ with the property that $p \Vdash^*_{\mathbb{P}} \exists x \ \varphi(x, \tau_0, \ldots, \tau_{n-1})$ holds, then there is $\sigma \in \mathrm{V}^{\mathbb{P}}$ with the property that $p \Vdash^*_{\mathbb{P}} \varphi(\sigma, \tau_0, \ldots, \tau_{n-1})$ holds (Hint: Set

$$D \ = \ \{q \leq_{\mathbb{P}} p \mid \exists \rho \in \mathrm{V}^{\mathbb{P}} \ q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \ldots, \tau_{n-1})\}.$$

Let $\langle a_\alpha \mid \alpha < \lambda \rangle$ enumerate a maximal antichain in D and pick a sequence $\langle \rho_\alpha \in \mathrm{V}^{\mathbb{P}} \mid \alpha < \lambda \rangle$ with $a_\alpha \Vdash^*_{\mathbb{P}} \varphi(\rho_\alpha, \tau_0, \ldots, \tau_{n-1})$ for every $\alpha < \lambda$. Use these sequences to construct a name σ with the desired properties).

Problem 44 (20 Bonus Points)**.** Let $\mathbb{B} = \langle \mathbb{B}, \leq, \wedge, \vee, 0, 1, ' \rangle$ be a complete boolean algebra and let $\mathbb{B}^*$ denote the corresponding partial order (see Problem 39). By induction on the structure of $\mathcal{L}_\in$-formulas, we define $[\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \in \mathbb{B}$ for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$.

(i) By simultaneous induction on the well-founded relation "$a \in \mathrm{tc}(b)$", we define

$$[\![\ \text{"}\tau_0 \in \tau_1\text{"}\ ]\!]_\mathbb{B} \ = \ \sup_\mathbb{B}\{r \wedge [\![\ \text{"}\tau_0 = \rho\text{"}\ ]\!]_\mathbb{B} \mid (\rho, r) \in \tau_1\}$$

and

$$[\![\ \text{"}\tau_0 = \tau_1\text{"}\ ]\!]_\mathbb{B} \ = \ \bigwedge_{i<2} \inf_\mathbb{B}\{\ r' \vee [\![\ \text{"}\rho \in \tau_i\text{"}\ ]\!]_\mathbb{B} \mid (\rho, r) \in \tau_{1-i}\}$$

(ii) $[\![\neg\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \ = \ [\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B}'$.

(iii) $[\![\varphi_0(\tau_0, \ldots, \tau_{n-1}) \wedge \varphi_1(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \ = \ [\![\varphi_0(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \wedge [\![\varphi_1(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B}$.

(iv) $[\![\exists x\ \varphi(x, \tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \ = \ \sup_\mathbb{B}\{[\![\varphi(\rho, \tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \mid \rho \in \mathrm{V}^{\mathbb{B}^*}\}$.

(1) Prove that the equivalence

$$p \Vdash^*_{\mathbb{B}^*} \varphi(\tau_0, \ldots, \tau_{n-1}) \iff p \leq [\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B}$$

holds for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$, $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$ and $p \in \mathbb{B}^*$.

(2) Prove that $\mathrm{V}^{\mathbb{B}^*}$ *is full*, i.e. for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_n)$ and all $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$, there is $\sigma \in \mathrm{V}^{\mathbb{B}^*}$ with

$$[\![\exists x\ \varphi(x, \tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B} \ = \ [\![\varphi(\sigma, \tau_0, \ldots, \tau_{n-1})]\!]_\mathbb{B}.$$

(3) Prove that the following statements hold for all $\tau_0, \tau_1, \tau_2 \in \mathrm{V}^{\mathbb{B}^*}$.

(a) $[\![\ \text{"}\tau_0 = \tau_0\text{"}\ ]\!]_\mathbb{B} \ = \ 1$.

(b) $[\![\ \text{"}\tau_0 = \tau_1\text{"}\ ]\!]_\mathbb{B} \ = \ [\![\ \text{"}\tau_1 = \tau_0\text{"}\ ]\!]_\mathbb{B}$.

(c) $[\![\ \text{"}\tau_0 = \tau_1\text{"}\ ]\!]_\mathbb{B} \cdot [\![\ \text{"}\tau_1 = \tau_2\text{"}\ ]\!]_\mathbb{B} \ \leq \ [\![\ \text{"}\tau_0 = \tau_2\text{"}\ ]\!]_\mathbb{B}$.

(d) $[\![\ \text{"}\tau_0 \in \tau_1\text{"}\ ]\!]_\mathbb{B} \cdot [\![\ \text{"}\tau_0 = \tau_2\text{"}\ ]\!]_\mathbb{B} \ \leq \ [\![\ \text{"}\tau_2 \in \tau_1\text{"}\ ]\!]_\mathbb{B}$.

(e) $[\![\ \text{"}\tau_0 \in \tau_1\text{"}\ ]\!]_\mathbb{B} \cdot [\![\ \text{"}\tau_1 = \tau_2\text{"}\ ]\!]_\mathbb{B} \ \leq \ [\![\ \text{"}\tau_0 \in \tau_2\text{"}\ ]\!]_\mathbb{B}$.

(4) Prove that $[\![(\mathit{Extensionality})]\!]_\mathbb{B} \ = \ 1$.

(5) Given a Σ_0-formula $\varphi(v_0, \ldots, v_{n-1})$, prove that

$$\varphi(a_0, \ldots, a_{n-1}) \longleftrightarrow [\![\varphi(\check{a}_0, \ldots, \check{a}_{n-1})]\!]_\mathbb{B} = 1$$

holds for all $a_0, \ldots, a_{n-1}$.

(6) Given $\tau \in \mathrm{V}^{\mathbb{B}^*}$, we have

$$[\![\ \text{"}\tau \in \mathrm{Ord}\text{"}\ ]\!]_\mathbb{B} \ = \ \sup_\mathbb{B}\{[\![\tau = \check{\alpha}]\!]_\mathbb{B} \mid \alpha \in \mathrm{Ord}\}.$$

(7) Prove that $[\![(\mathit{Infinity})]\!]_\mathbb{B} \ = \ 1$.

(8) Prove that $[\![\ \varphi\ ]\!]_\mathbb{B} \ = \ 1$ whenever φ is an axiom of ZFC.

Please hand in your solutions on Monday, June 24, before the lecture.