Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke | Problem sheet 10

Problem 37. Let M be a transitive model of ZFC, let $\mathbb{P}, \mathbb{Q} \in M$ be partial orders and let $\pi : \mathbb{Q} \longrightarrow \mathbb{P}$ be a dense embedding of partial orders that is an element of M. Prove the following statements:

(1) (1 Points) If G_0 and G_1 are $\mathbb{P}$-generic over M with $G_0 \subseteq G_1$, then $G_0 = G_1$.

(2) (1 Points) If G is $\mathbb{P}$-generic over M, then there is H $\mathbb{Q}$-generic over M with $M[G] = M[H]$.

(3) (1 Points) If H is $\mathbb{Q}$-generic over M, then there is G $\mathbb{P}$-generic over M with $M[G] = M[H]$.

(Hint: Use Problem 32 and the minimality of generic extensions).

Problem 38. Prove the following statements:

(1) (2 Points) If ZFC is consistent, then there is no Σ_1-formula $\varphi(v)$ such that

$$\mathrm{ZFC} \vdash \forall x \ [\varphi(x) \ \longleftrightarrow \ \text{"}x \textit{ is a cardinal}\text{"}].$$

(2) (2 Points) If ZFC is consistent, then there is no Σ_1-formula $\varphi(v_0, v_1)$ such that

$$\mathrm{ZFC} \vdash \forall x, y \ [\varphi(x, y) \ \longleftrightarrow \ \text{"}x = \mathcal{P}(y)\text{"}.$$

Problem 39. Let $X = (X, \tau)$ be a non-empty topological space. We let $\mathrm{ro}(X)$ denote the set of all regular open subsets of X (i.e. $int(cl(A)) = A$). Given $U, V \in \mathrm{ro}(X)$, define

$$U \vee V \ = \ int(cl(U \cup V))$$

and

$$U' \ = \ int(X \setminus U).$$

(1) (3 Points) Show that

$$\mathbb{B}(X) \ = \ \langle \mathrm{ro}(X), \subseteq, \cap, \vee, \emptyset, X, ' \ \rangle$$

is a *complete boolean algebra.*

Given a partial order $\mathbb{P}$, we define $\tau_{\mathbb{P}}$ to be the set of all subsets of $\mathbb{P}$ that are open in $\mathbb{P}$.

(2) (1 Points) Show that $X_{\mathbb{P}} = (\mathbb{P}, \tau_{\mathbb{P}})$ is a topological space.

Given a boolean algebra $\mathbb{B} = \langle B, \leq, \wedge, \vee, 0, 1, ' \ \rangle$, we define $\mathbb{B}^*$ to be the partial order $\langle B \setminus \{0\}, \leq \rangle$.

(3) (3 Points) Show that the map

$$\pi_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathrm{ro}(X_{\mathbb{P}}); \ p \longmapsto int(cl(\{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}))$$

is a dense embedding of $\mathbb{P}$ into the partial order $\mathbb{B}(X_{\mathbb{P}})^*$.

Problem 40. A partial order $\mathbb{P}$ is *separative* if for all conditions p and q in $\mathbb{P}$ with $p \not\leq_{\mathbb{P}} q$ there is a condition r in $\mathbb{P}$ with $r \leq_{\mathbb{P}} p$ and $q \perp_{\mathbb{P}} r$. Prove the following statements:

(1) (2 Points) If $\mathbb{B}$ is a boolean algebra, then $\mathbb{B}^*$ is separative.

(2) (2 Points) Show that a partial order $\mathbb{P}$ is separative if and only if the following statements hold:

  (a) The embedding $\pi_{\mathbb{P}}$ constructed in (3) of Problem 39 is injective.

  (b) $\forall p, q \in \mathbb{P}\ [p \leq_{\mathbb{P}} q \longleftrightarrow \pi_{\mathbb{P}}(p) \subseteq \pi_{\mathbb{P}}(q)]$.

(3) (2 Points) If $\mathbb{P}$ is a partial order, then there is a surjective complete embedding of $\mathbb{P}$ into a separative partial order (Hint: Show that

$$p \approx_{sep} q \iff \forall r \in \mathbb{P}\ [\ p \parallel_{\mathbb{P}} r \longleftrightarrow q \parallel_{\mathbb{P}} r\ ]$$

defines an equivalence relation on $\mathbb{P}$. Then define a suitable ordering of the set of $\approx_{sep}$-equivalence classes).

Please hand in your solutions on Monday, June 17, before the lecture.