Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke | Problem sheet 9

Problem 33. Remember that a subset of a topological space is *nowhere dense* if its closure has empty interior. Moreover, recall that the *Cantor space* is the topological space consisting of the set ${}^\omega 2$ of all functions from ω to 2 equipped with the topology whose basic open sets are of the form

$$B_s \;=\; \{x \in {}^\omega 2 \mid s \subseteq x\}$$

for some functions s contained in the set ${}^{<\omega}2$ of all functions $t : n \longrightarrow 2$ with $n < \omega$. We say that a sequence $\vec{s} = \langle s_i \in {}^{<\omega}2 \mid i < \omega\rangle$ is a *code for a closed nowhere dense subset of the Cantor space* if

$$N_{\vec{s}} \;=\; {}^\omega 2 \setminus \bigcup\{B_{s_i} \mid i < \omega\}$$

is a nowhere dense subset of ${}^\omega 2$.

(1) (2 Points) Prove that the statement

"*$\vec{s}$ is a code for a closed nowhere dense subset of the Cantor space*"

is absolute between transitive models of ZFC.

(2) (3 Points) Prove that the following statements are equivalent for every transitive model M of ZFC and every filter G on Cohen forcing $\mathbb{C}$:

(a) G is $\mathbb{C}$-generic over M.

(b) If $\vec{s} \in M$ is a code for a closed nowhere dense subset of the Cantor space, then $\bigcup G \notin N_{\vec{s}}$.

Problem 34 (5 Points)**.** Prove Lemma 4.2.4. from the lecture course: Assume ZF^- and let $\varphi(v_0, \ldots, v_{n-1})$ be an $\mathcal{L}_\in$-formula. The following statements are equivalent for every partial order $\mathbb{P}$, every $p \in \mathbb{P}$ and all $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$.

(1) $p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_{n-1})$.

(2) $q \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_{n-1})$ for every $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p$.

(3) The set $\{q \in \mathbb{P} \mid q \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_{n-1})\}$ is dense below p in $\mathbb{P}$.

Problem 35. Assume ZF^- and let $\varphi(v_0, \ldots, v_n)$ be an $\mathcal{L}_\in$-formula. If $\mathbb{P}$ is a partial order, $p \in \mathbb{P}$ and $\tau, \tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$, then the following statements hold:

(1) (2 Points) $p \Vdash^*_{\mathbb{P}} \forall x \; \varphi(x, \tau_0, \ldots, \tau_{n-1})$ if and only if $p \Vdash^*_{\mathbb{P}} \varphi(\sigma, \tau_0, \ldots, \tau_{n-1})$ for all $\sigma \in \mathrm{V}^{\mathbb{P}}$.

(2) (2 Points) $p \Vdash^*_{\mathbb{P}} \exists x \in \tau \; \varphi(x, \tau_0, \ldots, \tau_{n-1})$ if and only if the set

$$\{q \in \mathbb{P} \mid \exists (\rho, r) \in \tau \; [q \leq_{\mathbb{P}} r \wedge q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \ldots, \tau_{n-1})]\}$$

is dense below p in $\mathbb{P}$.

(3) (2 Points) $p \Vdash^*_{\mathbb{P}} \forall x \in \tau\, \varphi(x, \tau_0, \ldots, \tau_{n-1})$ if and only if $q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \ldots, \tau_{n-1})$ holds for all $(\rho, r) \in \tau$ and $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p, r$.

Problem 36 (6 Points)**.** Let $\mathrm{Fn}(\omega, \omega, \omega)$ denote the partial order consisting of all finite partial functions $p : \omega \xrightarrow{par} \omega$ ordered by reversed inclusion.

(1) (2 Points) Construct a dense subset D of the Cohen forcing $\mathbb{C}$ and a dense embedding of the partial order $(D, \supseteq)$ into $\mathrm{Fn}(\omega, \omega, \omega)$.

(2) (2 Points) Let $\mathbb{P}$ be a countable atomless partial order with a maximal element $\mathbb{1}_{\mathbb{P}}$. Prove that there is a dense embedding of a dense subset of $\mathrm{Fn}(\omega, \omega, \omega)$ into $\mathbb{P}$ (Hint: *Use a previous exercise to show that there is an infinite antichain below every condition in* $\mathbb{P}$*. Fix an enumeration* $\langle p_n \mid n < \omega \rangle$ *of* $\mathbb{P}$ *and define a function* π *by recursion. Set* $\pi(\emptyset) = \mathbb{1}_{\mathbb{P}}$*. If* $\pi(s)$ *is defined for some* $s : n \longrightarrow \omega$*, then extend* π *in a way such that* $\{\pi(s^\frown\langle m \rangle) \mid m < \omega\}$ *is a maximal antichain below* $\pi(s)$ *in* $\mathbb{P}$*. Moreover, if the conditions* p_n *and* $\pi(s)$ *are compatible in* $\mathbb{P}$*, then ensure that there is an* $m < \omega$ *with* $\pi(s^\frown\langle m \rangle) \leq_{\mathbb{P}} p_n$*. Show that the resulting function is a dense embedding.*).

Please hand in your solutions on Monday, June 03, before the lecture.