Models of Set Theory I. - Summer 2019

Problem 29 (5 Points)**.** Prove Lemma 4.1.7. from the lecture course: Let M be a transitive model of ZF^-, let $\mathbb{P} \in M$ be a partial order and let G be a filter on $\mathbb{P}$. Then the following statements are equivalent:

(1) G is $\mathbb{P}$-generic over M.
(2) $G \cap U \neq \emptyset$, whenever $U \in M$ is a dense open subset of $\mathbb{P}$.
(3) $D \cap G \neq \emptyset$, whenever $D \in M$ is a subset of $\mathbb{P}$ that is dense below some condition $p \in G$.

Moreover, if ZFC^M holds, then these statements are also equivalent to the following statements:

(4) $A \cap G \neq \emptyset$, whenever $A \in M$ is a maximal antichain in $\mathbb{P}$.
(5) $A \cap G \neq \emptyset$, whenever $A \in M$ is a maximal antichain in a dense subset $D \in M$ of $\mathbb{P}$, i.e. A is a maximal antichain in the partial order $(D, \leq_{\mathbb{P}} \restriction (D \times D))$.

Problem 30 (4 Points)**.** Let Φ be the Π_2-sentence given by Theorem 3.3.1. Show that there is a countable transitive set M with $M \cap \mathrm{Ord} \in \mathrm{Lim}$ and $(\neg\Phi)^M$.

Problem 31.

(1) (2 Points) Explicitly construct an infinite antichain in the Cohen forcing $\mathbb{C}$.
(2) (2 Points) Prove that every atomless partial order contains an infinite antichain.

Problem 32. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. We say that a function $\pi : \mathbb{Q} \to \mathbb{P}$ is a *complete embedding* if the following statements hold.

(a) If $q_0, q_1 \in \mathbb{Q}$ with $q_1 \leq_{\mathbb{Q}} q_0$, then $\pi(q_1) \leq_{\mathbb{P}} \pi(q_0)$.
(b) Given $q_0, q_1 \in \mathbb{Q}$, the conditions q_0 and q_1 are incompatible in $\mathbb{Q}$ if and only if the conditions $\pi(q_0)$ and $\pi(q_1)$ are incompatible in $\mathbb{P}$.
(c) The pointwise image of every maximal antichain in $\mathbb{Q}$ under π is a maximal antichain in $\mathbb{P}$.

We say that a function $\pi : \mathbb{Q} \to \mathbb{P}$ is a *dense embedding* if the above statements (a) and (b) hold and the image of $\mathbb{Q}$ under π is a dense subset of $\mathbb{P}$.

Prove the following statements:

(1) (2 Points) Every dense embedding is a complete embedding.
(2) (1 Point) Every partial order $\mathbb{Q}$ densely embeds into a partial order $\mathbb{P}$ with a maximal element $\mathbb{1}_{\mathbb{P}}$.

In the following, let M be a transitive model of ZFC, let $\mathbb{P}, \mathbb{Q} \in M$ be partial orders and let $\pi : \mathbb{Q} \longrightarrow \mathbb{P}$ be a function contained in M.

(3) (2 Points) If π is a complete embedding in M and G is $\mathbb{P}$-generic over M, then the preimage of G under π is a filter on $\mathbb{Q}$ that is $\mathbb{Q}$-generic over M.

(4) (2 Points) If π is a dense embedding in M and H is $\mathbb{Q}$-generic over M, then the set

$$\pi[H] \;=\; \{p \in \mathbb{P} \mid \exists q \in H \ \pi(q) \leq_{\mathbb{P}} p\}$$

is a filter on $\mathbb{P}$ that is $\mathbb{P}$-generic over M.

Please hand in your solutions on Monday, May 27, before the lecture.