Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke Problem sheet 7

Problem 25 (4 Points)**.** Given an uncountable regular cardinal κ, compute the order-type of $(\mathrm{L}_\kappa, <_{\mathrm{L}}\restriction (\mathrm{L}_\kappa \times \mathrm{L}_\kappa))$.

Problem 26. Let κ be an uncountable regular cardinal.

(a) The cardinal κ is *weakly Mahlo* if the set of all regular cardinals smaller than κ is a stationary subset of κ.

(b) The cardinal κ is *Mahlo* if the set of all inaccessible cardinals smaller than κ is a stationary subset of κ.

(1) (2 Points) Show that a cardinal is Mahlo if and only if it is inaccessible and weakly Mahlo.

(2) (2 Points) Show that $(\kappa \textit{ is Mahlo})^{\mathrm{L}}$ holds for all weakly Mahlo cardinals κ.

(3) (2 Points) Prove that the following statements are equivalent:

  (i) The theory ZFC + "*there is a strongly inaccessible cardinal*" is inconsistent.

  (ii) The theory ZFC+"*there is a weakly Mahlo cardinal*" is consistent relative to the theory ZFC + "*there is a strongly inaccessible cardinal*".

Problem 27. Given non-empty sets M and N with $M \subseteq N$, we let $M \preceq N$ ("*M is an elementary submodel of N*") denote the statement that

$$\mathsf{Sat}(M, a, k) \longleftrightarrow \mathsf{Sat}(N, a, k)$$

holds for all $k \in \mathsf{Fml}$ and all functions $a : n \longrightarrow M$ with $n < \omega$.

(1) (*The Tarski–Vaught–Test*, 2 Points) Prove that the following statements are equivalent for all sets $\emptyset \neq M \subseteq N$:

  (a) $M \preceq N$.

  (b) If $i, n < \omega$, $k \in \mathsf{Fml}$, $a : n \longrightarrow M$ and $y \in N$ with $\mathsf{Sat}(N, a|^i_y, k)$, then there is $z \in M$ with $\mathsf{Sat}(M, a|^i_z, k)$.

(2) (*Elementary Chains*, 2 Points) Let $\lambda \in \mathrm{Lim}$ and let $(M_\alpha \mid \alpha < \lambda)$ be a sequence of non-empty sets with $M_\alpha \preceq M_\beta$ for all $\alpha \leq \beta < \lambda$. Set $M = \bigcup\{M_\alpha \mid \alpha < \lambda\}$. Prove that $M_\alpha \preceq M$ holds for all $\alpha < \lambda$.

Problem 28. (1) (2 Points) Given an uncountable regular cardinal κ and a cardinal $\theta > \kappa$, construct a sequence $(M_\alpha \mid \alpha < \kappa)$ of subsets of $\mathrm{H}(\theta)$ such that the following statements hold for all $\beta < \kappa$:

  (a) $\kappa \in M_\beta$, $\beta \leq M_\beta \cap \kappa \in \kappa$ and $|M_\alpha| < \kappa$.

(b) If $\alpha < \beta$, then $M_\alpha \subsetneq M_\beta$ and $M_\alpha \preceq M_\beta \preceq \mathrm{H}(\theta)$.

(c) If $\beta \in \mathrm{Lim}$, then $M_\beta = \bigcup\{M_\alpha \mid \alpha < \beta\}$.

(Hint: Use Theorem 1.3.12 and Problem 27).

(2) (4 Points) Assume $\mathrm{V} = \mathrm{L}$. Let C be a closed unbounded subset of $\aleph_1$ and let S be a stationary subset of $\aleph_1$. Prove that there is an $\alpha \in \mathrm{Ord}$ with $(\mathrm{ZFC}^- + \text{"}\aleph_1 \textit{ exists }\text{"})^{\mathrm{L}_\alpha}$ and $\aleph_1^{\mathrm{L}_\alpha} \in C \cap S$ (Hint: Use the first part of the exercise and Theorem 3.3.1).

Please hand in your solutions on Monday, May 20, before the lecture.