Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke Problem sheet 4

Problem 13. (1) (2 Points) Construct an $\mathcal{L}_\in$-sentence φ with

$$\mathrm{ZFC} \vdash \forall \alpha \in \mathrm{Ord}\ [\varphi^{V_\alpha} \longrightarrow \text{"}\alpha \textit{ is a strong limit cardinal}\text{"}].$$

(2) (4 Points) Prove that the following statements are equivalent:

(a) ZFC $\vdash$ "*There are no strongly inaccessible cardinals*".

(b) There is an $\mathcal{L}_\in$-sentence φ with

$$\mathrm{ZFC} \vdash \forall \alpha \in \mathrm{Ord}\ [\varphi^{V_\alpha} \longleftrightarrow \text{"}\alpha \textit{ is strongly inaccessible}\text{"}].$$

(3) (4 Points) Show that the following statements are equivalent for every $\mathcal{L}_\in$-theory T extending ZFC and every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$:

(a) φ is a Σ_2^{T}-formula.

(b) There is an $\mathcal{L}_\in$-formula $\psi(v_0, \ldots, v_{n-1})$ with

$$\mathsf{T} \vdash \forall x_0, \ldots, x_{n-1}\ \big[\varphi(x_0, \ldots, x_{n-1}) \\ \longleftrightarrow \exists \alpha \in \mathrm{Ord}\ \big(x_0, \ldots, x_{n-1} \in V_\alpha \wedge \psi^{V_\alpha}(x_0, \ldots, x_{n-1})\big)\big].$$

(Hint: *Use (1) and the Σ_1-Reflection Principle*).

Problem 14 (2 Points)**.** Prove Proposition 1.3.2: Assume ZF$^-$. Then the relation $\prec_*$ defined by

$$a \prec_* b \iff \exists c, d\ [b = (c, a, d) \ \vee \ b = (c, d, a)]$$

is strongly well-founded.

Problem 15 (4 Points)**.** Prove Lemma 1.3.9: Assume ZF$^-$. Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$, if M is a non-empty set and $a : n \longrightarrow M$, then

$$\mathsf{Sat}(M, a, \ulcorner\varphi\urcorner) \longleftrightarrow \varphi^M(a(0), \ldots, a(n-1)).$$

Problem 16 (4 Points)**.** Show that if ZFC is consistent, then there is no $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$ with

$$\mathrm{ZFC} \ \vdash \ \forall x, y\ \forall k \in \mathsf{Fml}\ [(\varphi(k, x) \wedge \varphi(k, y)) \ \longrightarrow \ x = y]$$

and

$$\mathrm{ZFC} \vdash (\exists x\, \forall y\ [\psi(y) \longleftrightarrow x = y]) \longrightarrow \forall y\ [\varphi(\ulcorner \psi \urcorner, y) \longleftrightarrow \psi(y)]$$

for every $\mathcal{L}_\in$-formula $\psi(v)$.

Please hand in your solutions on Monday, April 29, before the lecture.