Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke | Problem sheet 1

Problem 1 (6 Points)**.** Recall the definitions of $\models$ and $\vdash$ and the basic properties of these relations from the *Introduction to Mathematical Logic* lecture course.

Problem 2 (4 Points)**.** Prove Lemma 1.1.2. from the lecture course: Let $\mathcal{L} \subseteq \mathcal{L}_*$ be first-order languages, let $\mathcal{M}$ be an $\mathcal{L}_*$-structure, let $a_1, \ldots, a_n \in |\mathcal{M}|$ and let $\psi(v_0, \ldots, v_n)$ be an $\mathcal{L}_*$-formula with the following properties:

(a) $\mathcal{M} \models \exists x \; \psi(\frac{x}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n})$.

(b) $\mathcal{M} \models \psi(\frac{c}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n})$ for every constant symbol c in $\mathcal{L}$.

(c) If f is an $(m+1)$-ary function symbom in $\mathcal{L}$, then

$$\mathcal{M} \models \forall x_0 \ldots x_m \; \Big[\left(\psi \left(\frac{x_0}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n} \right) \wedge \; \ldots \; \wedge \psi \left(\frac{x_m}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n} \right) \right) \longrightarrow \psi \left(\frac{f(x_0, \ldots, x_m)}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n} \right) \Big].$$

Then there exists a unique $\mathcal{L}$-substructure $\mathcal{N}$ of $\mathcal{M}$ with the following properties:

(1) $|\mathcal{N}| = \left\{ a \in |\mathcal{M}| \mid \mathcal{M} \models \psi \left(\frac{a}{v_0} \frac{a_1}{v_1} \ldots \frac{a_n}{v_n} \right) \right\}$.

(2) If $\varphi(w_0, \ldots, w_{m-1})$ is an $\mathcal{L}$-formula with $v_i \neq w_j$ for all $i \leq n$ and $j < m$ and $b_0, \ldots, b_{m-1} \in |\mathcal{N}|$, then

$$\mathcal{N} \models \varphi \left(\frac{b_0}{w_0} \ldots \frac{b_{m-1}}{w_{m-1}} \right) \iff \mathcal{M} \models \varphi^\psi \left(\frac{a_1}{v_1} \ldots \frac{a_n}{v_n} \frac{b_0}{w_0} \ldots \frac{b_{m-1}}{w_{m-1}} \right).$$

Problem 3 (8 points)**.** Prove Lemma 1.1.3. from the lecture course: Let $\mathcal{L} \subseteq \mathcal{L}_*$ be first-order languages. Given a collection Γ of $\mathcal{L}$-formulas, an $\mathcal{L}$-formula φ and an $\mathcal{L}_*$-formula $\psi(v_0, \ldots, v_n)$ with $v_i \notin \mathrm{free}(\Gamma \cup \{\varphi\})$ for all $i \leq n$, we define

$$\begin{aligned}
\Gamma^{\psi,\varphi} \; &= \Gamma^{\psi} \;\cup\; \Big\{\psi\left(\frac{w}{v_0}\right) \;\Big|\; w \in \mathrm{free}(\Gamma \cup \{\varphi\})\Big\} \;\cup\; \Big\{\exists x\; \psi\left(\frac{x}{v_0}\right)\Big\} \\
&\cup\; \Big\{\psi\left(\frac{c}{v_0}\right) \;\Big|\; c \text{ constant symbol in } \mathcal{L}\Big\} \\
&\cup\; \Big\{\forall x_0, \ldots, x_m \,\Big[\Big(\psi\left(\frac{x_0}{v_0}\right) \;\wedge \ldots \wedge\; \psi\left(\frac{x_0}{v_0}\right)\Big) \\
&\qquad \longrightarrow \psi\left(\frac{f(x_0, \ldots, x_m)}{v_0}\right)\Big] \;\Big|\; f \;(m+1)\text{-ary function symbol in } \mathcal{L}\Big\}.
\end{aligned}$$

If Γ is a collection of $\mathcal{L}$-formulas, φ is an $\mathcal{L}$-formula with $\Gamma \vdash \varphi$ and $\psi(v_0, \ldots, v_n)$ is an $\mathcal{L}_*$-formula with $v_i \notin \mathrm{free}(\Gamma \cup \{\varphi\})$ for all $i \leq n$, then $\Gamma^{\psi,\varphi} \vdash \varphi^{\psi}$.

Problem 4 (4 Points)**.** In the setting of the previsous exercise, show that it is in general not true that $\Gamma \vdash \varphi$ implies $\Gamma^{\psi} \vdash \varphi^{\psi}$.

Please hand in your solutions on Monday, April 08, before the lecture.