Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke Problem sheet 10

Problem 34 (4 Points)**.** Prove Proposition 6.1.5. from the lecture course: *Let X be a Polish space and $1 \leq \alpha < \omega_1$.*

(1) *The classes $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$ and $\mathbf{\Delta}^0_\alpha(X)$ are closed under finite intersections, finite unions and continuous preimages.*
(2) *The class $\mathbf{\Sigma}^0_\alpha(X)$ is closed under countable unions.*
(3) *The class $\mathbf{\Pi}^0_\alpha(X)$ is closed under countable intersections.*
(4) *The class $\mathbf{\Delta}^0_\alpha(X)$ is closed under complements.*

Problem 35 (8 Points)**.** Given topological spaces X and Y, a function $f : X \longrightarrow Y$ is *Borel*, if the preimage of every open (or, equivalently every Borel) subset of Y is Borel in X. Show that the following statements are equivalent for every Polish space X and every subset A of X.

(1) $A \in \mathbf{\Sigma}^1_1(X)$.
(2) There is a Polish space Y, a closed subset C of Y and a continuous function $f : Y \longrightarrow X$ with $A = f[C]$.
(3) There is a Polish space Y, a Borel subset B of Y and a Borel function $f : Y \longrightarrow X$ with $A = f[B]$.

(Hint: *Show that the graph of a Borel function $f : Y \longrightarrow X$ between Polish spaces X and Y is a Borel subset of $X \times Y$ by fixing a countable basis $\langle U_n \mid n < \omega\rangle$ of the topology on Y and showing that*

$$f(x) = y \iff \forall n < \omega \ \left[y \in U_n \longrightarrow x \in f^{-1}[U_n]\right]$$

holds for all $(x, y) \in X \times Y$)

Problem 36 (12 Points)**.** Given a tree $\mathbb{T}$, define $\mathbb{P}_\mathbb{T}$ to be the partial order $(\mathbb{T}, \geq_\mathbb{T})$.

(1) Let κ be an uncountable regular cardinal and $\mathbb{T}$ be a κ-Souslin tree with the property that for every $s \in \mathbb{T}$ and every $lh_\mathbb{T}(s) < \alpha < \kappa$, there is $t \in \mathbb{T}(\alpha)$ with $s <_\mathbb{T} t$. Show that the partial order $\mathbb{P}_\mathbb{T}$ satisfies the κ-chain condition and is $<\kappa$-distributive.

Fix a non-atomic partial order $\mathbb{P}$ that satisfies the κ-chain condition and is $<\kappa$-distributive.

(2) Construct $T_\mathbb{P} \subseteq \mathbb{P}$ such that $\mathbb{T}_\mathbb{P} = (T_\mathbb{P}, \geq_\mathbb{P})$ is a tree of height κ with the property that $\mathbb{T}_\mathbb{P}(\alpha)$ is a maximal antichain in $\mathbb{P}$ for every $\alpha < \kappa$ and the property that for every $s \in \mathbb{T}_\mathbb{P}$ and every $lh_{\mathbb{T}_\mathbb{P}}(s) < \alpha < \kappa$, there is $t \in \mathbb{T}_\mathbb{P}(\alpha)$ with $s <_{\mathbb{T}_\mathbb{P}} t$. (Hint: *Construct $\mathbb{T}(\alpha)$ recursively. If $\mathbb{T}(\alpha)$ is already constructed, then choose a maximal antichain A_p below p in $\mathbb{P}$ for*

every $p \in \mathbb{T}(\alpha)$ *and define* $\mathbb{T}(\alpha+1) = \bigcup_{p \in \mathbb{T}(\alpha)} A_p$. *If* $\alpha \in \kappa \cap \mathrm{Lim}$ *and* $\mathbb{T}(\bar{\alpha})$ *is already constructed for all* $\bar{\alpha} < \kappa$*, then show that the set* $D_\alpha = \{q \in \mathbb{P} \mid \forall \bar{\alpha} < \alpha \exists p \in \mathbb{T}(\bar{\alpha})\ q \leq_{\mathbb{P}} p\}$ *is open dense in* $\mathbb{P}$ *and define* $\mathbb{T}(\alpha)$ *to be a maximal antichain in* D_α)

(3) Show that $\mathbb{T}_{\mathbb{P}}$ is a κ-Souslin tree with the property that the inclusion map $i : \mathbb{P}_{\mathbb{T}_{\mathbb{P}}} \longrightarrow \mathbb{P}$ is a complete embedding.

(4) Conclude that non-atomic partial orders that satisfy the θ-chain condition for some uncoutable regular cardinal θ are not $<\theta^+$-distributive.

Please hand in your solutions on Wednesday, July 01, before the lecture.