Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke Problem sheet 6

Problem 22 (4 Points)**.** Let $\varphi(v_0, \ldots, v_n)$ be an $\mathcal{L}_\in$-formula, $\mathbb{P}$ be a partial order, $p \in \mathbb{P}$ and $\sigma_0, \ldots, \sigma_n, \tau_0, \ldots, \tau_n \in \mathrm{V}^{\mathbb{P}}$ with

$$p \Vdash^*_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_n) \;\wedge\; \sigma_0 = \tau_0 \;\wedge \ldots \wedge \sigma_n = \tau_n.$$

Show that $p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \ldots, \tau_n)$.

Problem 23 (6 Points)**.** Prove the *Maximality Principle*: *Let $\varphi(v_0, \ldots, v_n)$ be an $\mathcal{L}_\in$-formula. If $\mathbb{P}$ is a partial order, $p \in \mathbb{P}$ and $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$ such that $p \Vdash^*_{\mathbb{P}} \exists x \; \varphi(x, \tau_0, \ldots, \tau_{n-1})$ holds, then there is $\sigma \in \mathrm{V}^{\mathbb{P}}$ with $p \Vdash^*_{\mathbb{P}} \varphi(\sigma, \tau_0, \ldots, \tau_{n-1})$* (*Hint*: Set $D = \{q \leq_{\mathbb{P}} p \mid \exists \rho \in \mathrm{V}^{\mathbb{P}} \; q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \ldots, \tau_{n-1})\}$. Let $\langle a_\alpha \mid \alpha < \lambda\rangle$ enumerate a maximal antichain in D and pick a sequence $\langle \rho_\alpha \in \mathrm{V}^{\mathbb{P}} \mid \alpha < \lambda\rangle$ with $a_\alpha \Vdash^*_{\mathbb{P}} \varphi(\rho_\alpha, \tau_0, \ldots, \tau_{n-1})$ for every $\alpha < \lambda$. Use these sequences to construct a name σ with the desired properties).

Problem 24 (14 Points + 16 Bonus Points)**.** Let $\mathbb{B} = \langle \mathbb{B}, \leq, \wedge, \vee, 0, 1, ' \rangle$ be a complete boolean algebra and let $\mathbb{B}^*$ denote the corresponding partial order (see Problem 20). By induction on the structure of $\mathcal{L}_\in$-formulas, we define $[\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \in \mathbb{B}$ for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$.

(i) By simultaneous induction on the well-founded relation "$a \in \mathrm{tc}(b)$", we define

$$[\![\text{"}\tau_0 \in \tau_1\text{"}]\!]_{\mathbb{B}} \;=\; \sup_{\mathbb{B}}\{r \wedge [\![\text{"}\tau_0 = \rho\text{"}]\!]_{\mathbb{B}} \mid (\rho, r) \in \tau_1\}$$

and

$$[\![\text{"}\tau_0 = \tau_1\text{"}]\!]_{\mathbb{B}} \;=\; \bigwedge_{i<2} \inf_{\mathbb{B}}\{ r' \vee [\![\text{"}\rho \in \tau_i\text{"}]\!]_{\mathbb{B}} \mid (\rho, r) \in \tau_{1-i}\}$$

(ii) $[\![\neg\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \;=\; [\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]'_{\mathbb{B}}$.

(iii) $[\![\varphi_0(\tau_0, \ldots, \tau_{n-1}) \wedge \varphi_1(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \;=\; [\![\varphi_0(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \wedge [\![\varphi_1(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}}$.

(iv) $[\![\exists x \; \varphi(x, \tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \;=\; \sup_{\mathbb{B}}\{[\![\varphi(\rho, \tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \mid \rho \in \mathrm{V}^{\mathbb{B}^*}\}$.

(1) Prove that the equivalence

$$p \Vdash^*_{\mathbb{B}^*} \varphi(\tau_0, \ldots, \tau_{n-1}) \iff p \leq [\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}}$$

holds for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$, $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$ and $p \in \mathbb{B}^*$.

(2) Prove that $\mathrm{V}^{\mathbb{B}^*}$ *is full*, i.e. for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_n)$ and all $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$, there is $\sigma \in \mathrm{V}^{\mathbb{B}^*}$ with

$$[\![\exists x \; \varphi(x, \tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \;=\; [\![\varphi(\sigma, \tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}}.$$

(3) Prove that the following statements hold for all $\tau_0, \tau_1, \tau_2 \in \mathrm{V}^{\mathbb{B}^*}$.

(a) $[\![\text{"}\tau_0 = \tau_0\text{"}]\!]_{\mathbb{B}} \;=\; 1$.

(b) $[\![\text{"}\tau_0 = \tau_1\text{"}]\!]_{\mathbb{B}} \;=\; [\![\text{"}\tau_1 = \tau_0\text{"}]\!]_{\mathbb{B}}$.

(c) $[\![\text{"}\tau_0 = \tau_1\text{"}]\!]_{\mathbb{B}} \cdot [\![\text{"}\tau_1 = \tau_2\text{"}]\!]_{\mathbb{B}} \;\leq\; [\![\text{"}\tau_0 = \tau_2\text{"}]\!]_{\mathbb{B}}$.

(d) $[\![\text{"}\tau_0 \in \tau_1\text{"}]\!]_{\mathbb{B}} \cdot [\![\text{"}\tau_0 = \tau_2\text{"}]\!]_{\mathbb{B}} \ \leq \ [\![\text{"}\tau_2 \in \tau_1\text{"}]\!]_{\mathbb{B}}$.

(e) $[\![\text{"}\tau_0 \in \tau_1\text{"}]\!]_{\mathbb{B}} \cdot [\![\text{"}\tau_1 = \tau_2\text{"}]\!]_{\mathbb{B}} \ \leq \ [\![\text{"}\tau_0 \in \tau_2\text{"}]\!]_{\mathbb{B}}$.

(4) Prove that $[\![(\textit{Extensionality})]\!]_{\mathbb{B}} \ = \ 1$.

(5) Given a Δ_0-formula $\varphi(v_0, \ldots, v_{n-1})$, prove that

$$\varphi(a_0, \ldots, a_{n-1}) \ \longleftrightarrow \ [\![\varphi(\check{a}_0, \ldots, \check{a}_{n-1})]\!]_{\mathbb{B}} = 1$$

holds for all $a_0, \ldots, a_{n-1}$.

(6) Given $\tau \in \mathrm{V}^{\mathbb{B}^*}$, we have

$$[\![\text{"}\tau \in \mathrm{Ord}\text{"}]\!]_{\mathbb{B}} \ = \ \sup_{\mathbb{B}}\{[\![\tau = \check{\alpha}]\!]_{\mathbb{B}} \mid \alpha \in \mathrm{Ord}\}.$$

(7) Prove that $[\![(\textit{Infinity})]\!]_{\mathbb{B}} \ = \ 1$.

(8) Prove that $[\![\ \varphi \ ]\!]_{\mathbb{B}} \ = \ 1$ whenever φ is an axiom of ZFC.

Please hand in your solutions on Wednesday, June 03, before the lecture.