Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke | Problem sheet 5

Problem 19 (6 Points)**.** Prove the following statements.

(1) If ZF is consistent, then there is no Σ-formula $\varphi(v)$ such that

$$\mathrm{ZFC} \vdash \forall x \; [\varphi(x) \longleftrightarrow \text{"}x \textit{ is a cardinal}\text{"}].$$

(2) If ZF is consistent, then there is no Σ-formula $\varphi(v_0, v_1)$ such that

$$\mathrm{ZFC} \vdash \forall x, y \; [\varphi(x, y) \longleftrightarrow \text{"}x = \mathcal{P}(y)\text{"}.$$

Problem 20 (10 Points)**.** Let (X, τ) be a non-empty topological space. We let $\mathrm{ro}(X, \tau)$ denote the set of all regular open subsets of X (i.e. $int(cl(A)) = A$). Define $U \vee V = int(cl(U \cup V))$ and $U' = int(X \setminus U)$ for all $U, V \in \mathrm{ro}(X, \tau)$.

(1) Show that

$$\mathbb{B}(X, \tau) \;=\; \langle \mathrm{ro}(X, \tau), \subseteq, \cap, \vee, \emptyset, X, ' \,\rangle$$

is a complete boolean algebra.

Given a partial order $\mathbb{P}$, we define $\tau_{\mathbb{P}}$ to be the set of all subsets of $\mathbb{P}$ that are open in $\mathbb{P}$.

(2) Show that $(\mathbb{P}, \tau_{\mathbb{P}})$ is a topological space.

Given a boolean algebra $\mathbb{B} = \langle B, \leq, \wedge, \vee, 0, 1, ' \,\rangle$, we define $\mathbb{B}^*$ to be the partial order $\langle B \setminus \{0\}, \leq, \rangle$.

(3) Show that the map

$$\pi_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathrm{ro}(\mathbb{P}, \tau_{\mathbb{P}}); \; p \longmapsto int(cl(\{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}))$$

is a dense embedding of $\mathbb{P}$ into the partial order $\mathbb{B}(\mathbb{P}, \tau_{\mathbb{P}})^*$.

Problem 21 (8 Points)**.** A partial order $\mathbb{P}$ is *separative* if for all conditions p and q in $\mathbb{P}$ with $p \not\leq_{\mathbb{P}} q$ there is a condition r in $\mathbb{P}$ with $r \leq_{\mathbb{P}} p$ and $q \perp_{\mathbb{P}} r$.

(1) Show: if $\mathbb{B}$ is a boolean algebra, then $\mathbb{B}^*$ is separative.

(2) Show that a partial order $\mathbb{P}$ is separative if and only if the following statements hold.

(a) The embedding $\pi_{\mathbb{P}}$ constructed in part (3) of Problem 20 is injective.

(b) $\forall p, q \in \mathbb{P} \; [p \leq_{\mathbb{P}} q \longleftrightarrow \pi_{\mathbb{P}}(p) \subseteq \pi_{\mathbb{P}}(q)$.

(3) If $\mathbb{P}$ is a partial order, then there is a surjective complete embedding of $\mathbb{P}$ into a separative partial order (Hint: *Show that*

$$p \approx_{sep} q \iff \forall r \in \mathbb{P}\ [\ p \parallel_{\mathbb{P}} r \longleftrightarrow q \parallel_{\mathbb{P}} r\ ]$$

defines an equivalence relation on $\mathbb{P}$*. Then define a suitable ordering of the quotient* $\mathbb{P}/\approx_{sep}$).

Please hand in your solutions on Wednesday, May 20, in room 4.004 at 12 pm.