

Models of Set Theory I. - Summer 2015

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Problem sheet 3

Problem 10 (2 Points). Assume ZF. Show that the following statements are equivalent.

- (1) $V = \text{HOD}$.
- (2) There is a well-ordering of V that is definable by an \mathcal{L}_\in -formula without parameters.

Problem 11 (7 Points). Prove Lemma 3.1.7. from the lecture course: *Let M be a transitive model of ZF^- , $\mathbb{P} \in M$ be a partial order and G be a filter on \mathbb{P} . Then the following statements are equivalent:*

- (1) G is \mathbb{P} -generic over M .
- (2) $G \cap U \neq \emptyset$, whenever $U \in M$ is a dense open subset of \mathbb{P} .
- (3) $D \cap G \neq \emptyset$, whenever $D \in M$ is a subset of \mathbb{P} that is dense below some condition $p \in G$.

Moreover, if ZFC^M holds, then these statements are also equivalent to the following statements:

- (4) $A \cap G \neq \emptyset$, whenever $A \in M$ is a maximal antichain in \mathbb{P} .
- (5) $A \cap G \neq \emptyset$, whenever $A \in M$ is a maximal antichain in a dense subset $D \in M$ of \mathbb{P} (i.e. A is a maximal antichain in the partial order $(D, \leq_{\mathbb{P}})$).

Problem 12 (4 Points).

- (1) Explicitly construct an infinite antichain in the Cohen forcing \mathbb{C} .
- (2) Prove that every atomless partial order contains an infinite antichain.

Problem 13 (10 Points). Let \mathbb{P} and \mathbb{Q} be partial orders. We say that a function $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a *complete embedding* if the following statements hold.

- (a) If $q_0, q_1 \in \mathbb{Q}$ with $q_1 \leq_{\mathbb{Q}} q_0$, then $\pi(q_1) \leq_{\mathbb{P}} \pi(q_0)$.
- (b) Given $q_0, q_1 \in \mathbb{Q}$, the conditions q_0 and q_1 are incompatible in \mathbb{Q} if and only if the conditions $\pi(q_0)$ and $\pi(q_1)$ are incompatible in \mathbb{P} .
- (c) The image of every maximal antichain in \mathbb{Q} under π is a maximal antichain in \mathbb{P} .

We say that a function $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ is a *dense embedding* if the above statements (a) and (b) hold and the image of \mathbb{Q} under π is a dense subset of \mathbb{P} .

Prove the following statements.

- (1) Every dense embedding is a complete embedding.

Let M be a transitive model of ZFC, $\mathbb{P}, \mathbb{Q} \in M$ be partial orders and $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ be a function contained in M .

- (2) If π is a complete embedding in M and G is \mathbb{P} -generic over M , then the preimage of G under π is a filter on \mathbb{Q} that is \mathbb{Q} -generic over M .
- (3) If π is a dense embedding in M and H is \mathbb{Q} -generic over M , then the set $\pi[H] = \{p \in \mathbb{P} \mid \exists q \in H \pi(q) \leq_{\mathbb{P}} p\}$ is a filter on \mathbb{P} that is \mathbb{P} -generic over M .

Problem 14 (3 Points). Show that every partial order \mathbb{Q} densely embeds into a partial order \mathbb{P} with a maximal element $\mathbb{1}_{\mathbb{P}}$.

Please hand in your solutions on Wednesday, May 06, before the lecture.