Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke | Problem sheet 1

Problem 1 (3 Points). Prove Proposition 1.1.2. from the lecture course: *If $\mathcal{M} = (M, E)$ is an $\mathcal{L}_\in$-structure, $W \subseteq M$ is a class in $\mathcal{M}$ and $\varphi(v_0, \dots, v_{n-1})$ is an $\mathcal{L}_\in$-formula, then*

$$\mathcal{M} \models \varphi^W(a_0, \dots, a_{n-1}) \iff (W, E \cap (W \times W)) \models \varphi(a_0, \dots, a_{n-1})$$

holds for all $a_0, \dots, a_{n-1} \in W$.

Problem 2 (4 Points). Prove Lemma 1.1.4.(ii) from the lecture course: *Assume* ZF^-. *Let W be a non-empty transitive class and $\varphi(v_0, \dots, v_{n+1})$ be an $\mathcal{L}_\in$-formula. Then $(Replacement_\varphi)^W$ holds if and only if for all $c_0, \dots, c_{n-1} \in W$, either there are $a, b_0, b_1 \in W$ with $\varphi^W(a, b_i, c_0, \dots, c_{n-1})$ for all $i < 2$, or there is an $a \in W$ with $\neg\varphi^W(a, b, c_0, \dots, c_{n-1})$ for all $b \in W$, or*

$$\{b \in W \mid \exists a \in d \ \varphi^W(a, b, c_0, \dots, c_{n-1})\} \in W$$

for every $d \in W$.

Problem 3 (2 Points). Prove Proposition 1.1.17.(ii) from the lecture course: *If T is an $\mathcal{L}_\in$-theorie with $T \vdash (Collection_\varphi)$ for every $\mathcal{L}_\in$-formula φ, then the classes of Σ^T_n-, Π^T_n- and Δ^T_n-formulas are closed under $\forall x \in v$ and $\exists x \in v$.*

Problem 4 (5 Points). Prove the Σ-*Recursion Theorem* (Theorem 1.1.18) from the lecture course: *Given an $\mathcal{L}_\in$-formula $\psi(v_0, \dots, v_{n+1})$ and Σ-formulas $\varphi_0(v_0, \dots, v_{n+2})$ and $\varphi_1(v_0, \dots, v_{n+1})$, there is a Σ-formula $\Phi(v_0, \dots, v_{n+1})$ such that the theory $\mathsf{ZF}^- - (Infinity)$ proves that for all $c_0, \dots, c_{n-1}$ with the property that $R = \{(a, b) \mid \psi(a, b, c_0, \dots, c_{n-1})\}$ is a strongly well-founded relation, $G = \{((a_0, a_1), b) \mid \varphi_0(a_0, a_1, b, c_0, \dots, c_{n-1})\}$ is a function with domain $\mathrm{V} \times \mathrm{V}$ and $P = \{(a, b) \mid \varphi_1(a, b, c_0, \dots, c_{n-1})\}$ is a function with domain V and $P(a) = \mathrm{pred}_R(a)$ for all a, the class $F = \{ (a, b) | \ \Phi(a, b, c_0, \dots, c_{n-1})\}$ is a function with domain V and $F(a) = G(a, F \restriction \mathrm{pred}_R(a))$ for all a.*

Problem 5 (6 points). Assume ZFC. Examine which ZFC-axioms hold in $\mathrm{H}(\kappa)$ for various infinite cardinals κ.

Please hand in your solutions on Wednesday, April 22, before the lecture.