Set theory - Winter Semester 2014

Problems	Dr. Philipp Schlicht
Series 13	Dr. Philipp Lücke

Problem 44. Let κ be an uncountable cardinal and $\mathbb{S}$ be a tree of height κ with $|\mathbb{S}(\alpha)| < \kappa$ for all $\alpha < \kappa$.

(1) (2 points) Show that there is a closed unbounded subset D of κ and a tree $\mathbb{T} = (\kappa, \leq_{\mathbb{T}})$ such that the following statements hold.
 (a) The trees $\mathbb{S}$ and $\mathbb{T}$ are isomorphic.
 (b) If $\langle d_\alpha \mid \alpha < \kappa \rangle$ is the monotone enumeration of D, then the α-th level of $\mathbb{T}$ is equal to the interval $[d_\alpha, d_{\alpha+1})$.

(2) (2 points) Let $\mathbb{T}$ be the tree constructed above. Show that there is a C-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that the following statements hold for every limit ordinal α less than κ.
 (a) If $\bar{\alpha} = \sup(D \cap \alpha) < \alpha$, then $C_\alpha = (\bar{\alpha}, \alpha)$.
 (b) If $\alpha = \sup(D \cap \alpha)$, then C_α is equal to the set $D_\alpha = \{\bar{\alpha} < \alpha \mid \bar{\alpha} <_{\mathbb{T}} \alpha\}$ together with the limit points of D_α smaller then α (i.e. C_α is the closure of D_α in the order-topology on α).

(3) (6 points) Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be the C-sequence constructed above. Assume that there is a closed unbounded subset C of κ such that for every $\alpha < \kappa$ there is a $\alpha \leq \beta < \kappa$ with $C \cap \alpha = C_\beta \cap \alpha$. Let Γ denote the set of all successor ordinals $\gamma < \kappa$ with $C \cap [d_\gamma, d_{\gamma+1}) \neq \emptyset$. Show that

$$c = \{\min(C \cap [d_\gamma, d_{\gamma+1})) \mid \gamma \in \Gamma\}$$

is a chain in $\mathbb{T}$ of order-type κ.

Problem 45 (6 points)**.** Show that the following statements are equivalent for every inaccessible cardinal κ.

(1) The cardinal κ has the tree property.

(2) If $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a C-sequence, then there is a club subset C of κ such that for every $\alpha < \kappa$ there is a $\alpha \leq \beta < \kappa$ with $C \cap \alpha = C_\beta \cap \alpha$.

(Hint: *Use Lemma 5.5.11 and Lemma 5.5.12 to prove the implication* (1) $\Rightarrow$ (2). *Use Problem 44 to prove the converse implication.*)

Due Wednesday, January 21, before the lecture.