Set theory - Winter Semester 2014

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Problem 43. We let $\mathbb{T} = (I, \subseteq)$ denote the tree consisting of all strictly increasing functions $f : \alpha \longrightarrow \mathbb{Q}$ with $\alpha < \omega_1$ ordered by inclusion (see Problem 37).

(1) (8 points) Construct a subset A of I with the following properties.
  (a) A is closed under initial segments, i.e. if $t \in A$ and $s \in I$ with $s \subseteq t$, then $s \in A$.
  (b) If $\alpha < \omega_1$, then the set $A \cap \mathbb{T}(\alpha)$ is countable.
  (c) If $\alpha \leq \beta < \omega_1$, $s \in A \cap \mathbb{T}(\alpha)$ and $p, q \in \mathbb{Q}$ such that $p < q$ and $s(\gamma) < p$ for all $\gamma < \alpha$, then there is $t \in A \cap \mathbb{T}(\beta)$ such that $s \subseteq t$ and $t(\gamma) < q$ for all $\gamma < \beta$.

  (Hint: *Define* $A \cap \mathbb{T}(\alpha)$ *by recursion on* $\alpha < \omega_1$).

(2) (4 points) Show that the resulting tree $\mathbb{A} = (A, \subseteq)$ is an Aronszajn tree.

(3) (4 points) Show that $\mathbb{A}$ is not a Souslin tree.

(4) (4 bonus points) Show that there is no $S \subseteq A$ such that $(S, \subseteq)$ is a Souslin tree.

Due Wednesday, January 14, before the lecture.