Set theory - Winter Semester 2014

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Problem 41 (16 points).

(1) Let $\mathbb{T}$ be a tree with $\mathrm{lh}_{\mathbb{T}}(b) < \omega$ for all $b \in \sigma\mathbb{T}$. Show that there is a unique function $\mathrm{rnk}_{\mathbb{T}} : \mathbb{T} \longrightarrow \mathrm{Ord}$ with

$$\mathrm{rnk}_{\mathbb{T}}(s) \;=\; \sup\{\mathrm{rnk}_{\mathbb{T}}(t) + 1 \mid t \in \mathrm{succ}_{\mathbb{T}}(s)\}$$

for all $s \in \mathbb{T}$. We define $\mathrm{rnk}(\mathbb{T}) = \mathrm{rnk}_{\mathbb{T}}(\mathrm{root}(\mathbb{T}))$ to be the *rank of* $\mathbb{T}$.

(2) Given $\alpha \in \mathrm{Ord}$, let $\mathbb{T}^{\alpha}$ denote the partial order consisting of all strictly decreasing functions $d : n \longrightarrow \alpha$ with $n < \omega$ ordered by inclusion. Then $\mathbb{T}^{\alpha}$ is a tree with $\mathrm{lh}_{\mathbb{T}^{\alpha}}(b) < \omega$ for all $b \in \sigma\mathbb{T}^{\alpha}$. Show that $\mathrm{rnk}(\mathbb{T}^{\alpha}) = \alpha$.

Given trees $\mathbb{S}$ and $\mathbb{T}$, we let $\mathbb{S} \preceq \mathbb{T}$ denote the statement that there is a function $e : \mathbb{S} \longrightarrow \mathbb{T}$ with

$$s <_{\mathbb{S}} t \;\longrightarrow\; e(s) <_{\mathbb{T}} e(t)$$

for all $s, t \in \mathbb{S}$.

(3) Let $\mathbb{S}$ and $\mathbb{T}$ be trees with $\mathbb{S} \preceq \mathbb{T}$ and $\mathrm{lh}_{\mathbb{T}}(b) < \omega$ for all $b \in \sigma\mathbb{T}$. Show that $\mathrm{lh}_{\mathbb{S}}(b) < \omega$ for all $b \in \sigma\mathbb{S}$ and $\mathrm{rnk}(\mathbb{S}) \leq \mathrm{rnk}(\mathbb{T})$.

(4) Given $\alpha \in \mathrm{Ord}$, show that $\sigma\mathbb{T}^{\alpha} \preceq \mathbb{T}^{\alpha+1}$ and $\mathbb{T}^{\alpha+1} \preceq \sigma\mathbb{T}^{\alpha}$ hold.

(5) Show that

$$\alpha \leq \beta \iff \mathbb{T}^{\alpha} \preceq \mathbb{T}^{\beta}$$

holds for all $\alpha, \beta \in \mathrm{Ord}$ *(Hint: Use the above results and Problem 40).*

Problem 42 (4 points).

(1) Show that every infinite sequence $\langle x_n \mid n < \omega \rangle$ of real numbers has an infinite subsequence that is either constant or strictly increasing or strictly decreasing.

(2) Construct a sequence $\langle x_{\alpha} \mid \alpha < \omega_1 \rangle$ of real numbers with the property that every uncountable subsequence is neither constant nor strictly increasing nor strictly decreasing.

Happy Holidays!

Due Wednesday, January 07, before the lecture.