Set theory - Winter Semester 2014

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Problem 26 (4 points)**.** Suppose that E is an equivalence relation on a set x. A *transversal* for E is a set $y \subseteq x$ such that

(1) for all $u \in x$, there is some $v \in y$ with uEv and
(2) for all $u, v \in y$ with $u \neq v$, $\neg uEv$.

Show that the following statement is equivalent to the Axiom of Choice: *For every set x and every equivalence relation E on x, there is a transversal for E.*

Problem 27 (2 points)**.** Show that every maximal filter on a set x is an ultrafilter on x.

Problem 28 (2 points)**.** A subset C of a partial order $(A, \leq)$ is called *cofinal* if

$$\forall x \in A \ \exists y \in C \ x \leq y$$

holds. Suppose that $\mu \leq \kappa$ are infinite cardinals. Let $[\kappa]^\mu = \{x \subseteq \kappa \mid |x| = \mu\}$. Show that $|[\kappa]^\mu| \leq |C| \cdot 2^\mu$ if C is cofinal in $([\kappa]^\mu, \subseteq)$.

Problem 29 (4 points)**.** Suppose that there are ω many people p_n for $n \in \omega$, standing in a line, such that p_m can see p_n for all $m < n$. Each p_n is wearing a hat with color red or blue which only p_m with $m < n$ can see. Everybody independently guesses the color of their own hat, and the game is won if all but finitely many guesses are right. The people can agree on a strategy before seeing the hats. Find a winning strategy using the Axiom of Choice. *(Hint: use the equivalence relation E_0 defined in the lecture.)*

Problem 30 (10 points)**.** Suppose that γ is a limit ordinal with $\mathrm{cof}(\gamma) > \omega$. Prove the following statements.

(1) $\kappa \setminus \mathrm{Lim}$ is not club in γ.
(2) If C, D are club in γ, then $C \cap D$ is club in γ.
(3) If $S \subseteq \gamma$ is unbounded, then the sets of limit points $\alpha < \gamma$ of S is club in γ.
(4) If γ is regular and $f\colon \gamma \to \gamma$ is a function, then set of *closure points* $\alpha < \gamma$ with $f[\alpha] \subseteq \alpha$ is club in γ.
(5) Suppose that G is a countable subgroup of the group $Sym(\omega_1)$ of permutations of ω_1. Show that the set of $\alpha < \omega_1$ such that $g[\alpha] \subseteq \alpha$ for all $g \in G$ is club in ω_1.

Due Wednesday, November 26, before the lecture.