PCF

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1 Good points

Convention. In the whole section, X will stand for a set of cardinality κ and I will denote an ideal on X. If not stated otherwise, $\vec{f} = \langle f_i \mid i < \lambda \rangle$ is an I-increasing sequence in ${}^X\mathrm{Ord}$. Furthermore, we call $\lambda = \mathrm{lh}(\vec{f})$ the **length** of $\vec{f}$. If $\alpha \leq \lambda$, then we write $\vec{f} \restriction \alpha$ for the sequence $\langle f_i \mid i < \alpha \rangle$.

Definition 1.1. Two $I-$increasing sequences $\vec{f}$ and $\vec{g}$ of functions in ${}^X\mathrm{Ord}$ are said to be **cofinally interleaved**, if

1. for all $i < \mathrm{lh}(\vec{f})$ there exists $j < \mathrm{lh}(\vec{g})$ such that $f_i <_I g_j$,
2. for all $j < \mathrm{lh}(\vec{g})$ there exists $i < \mathrm{lh}(\vec{f})$ such that $g_j <_I f_i$.

Remark 1.2. If $\vec{f}$ and $\vec{g}$ are cofinally interleaved, then every eub for $\vec{f}$ is an eub for $\vec{g}$ and vice versa. Conversely, if $\vec{f}$ and $\vec{g}$ have a common eub, then they are cofinally interleaved.

Remark 1.3. If $\vec{f}$ and $\vec{g}$ are cofinally interleaved, then $\mathrm{cf}(\mathrm{lh}(\vec{f})) = \mathrm{cf}(\mathrm{lh}(\vec{g}))$.

Proof. Assume that $\vec{f}$ and $\vec{g}$ are cofinally interleaved and let $\lambda = \mathrm{lh}(\vec{f})$ and $\mu = \mathrm{lh}(\vec{g})$. By symmetry it is enough to show that $\mathrm{cf}(\mu) \leq \mathrm{cf}(\lambda)$. Let

$$\pi : \lambda \to \mu, i \mapsto \min\{j < \mathrm{lh}(\vec{g}) \mid f_i <_I g_j\}.$$

By assumption, π is well-defined and it is cofinal in $\mathrm{lh}(\vec{g})$: Let $j < \mu$. Then there is $i < \lambda$ such that $g_j <_I f_i$ and $f_i <_I g_{\pi(i)}$. Then $j < \pi(i)$. This shows that $\mathrm{cf}(\mu) \leq \mathrm{cf}(\lambda)$. $\square$

Definition 1.4. Let $|X| = \kappa$ and I an ideal on X and $\vec{f}$ a $<_I$-increasing sequence of functions in ${}^X\mathrm{Ord}$. Then $\alpha \leq \mathrm{lh}(\vec{f})$ is said to be a **good** point for $\vec{f}$ if only if $\mathrm{cf}(\alpha) > \kappa$ and there is an eub h for $\vec{f} \restriction \alpha$ such that $\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$ for all $x \in X$.

Remark 1.5. By adjusting the eub h on an I-small set, in Definition 1.4 it is enough to assume that $\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$ for I-almost all $x \in X$.

Remark 1.6. If $\mathrm{cf}(\alpha) > \kappa$ and h is an eub for $\vec{f} \restriction \alpha$ and $\mu > \kappa$ is a cardinal such that $\mathrm{cf}(h(x)) = \mu$ for all $x \in X$, then $\mu = \mathrm{cf}(\alpha)$.

Proof. Let h be an eub for $\vec{f} \restriction \alpha$ such that $\mathrm{cf}(h(x)) = \mu$ for all $x \in X$. This means that for every $x \in X$ there is a strictly increasing sequence $\langle h_i(x) \mid i < \mu \rangle$ cofinal in $h(x)$. We show that $\vec{h} = \langle h_i \mid i < \mu \rangle$ is cofinally interleaved with $\vec{f} \restriction \alpha$. Then the claim follows from Remark 1.3.

Firstly, note that each $h_i <_I h$, so since h is an eub for $\vec{f} \restriction \alpha$, there is $j < \alpha$ such that $h_i <_I f_j$. Conversely, let $i < \alpha$. Without loss of generality (by adjusting f_i on an I-small set) we can assume that $f_i < h$. Then for every $x \in X$ there is $i_x < \mu$ such that $f_i(x) < h_{i_x}(x)$. Now consider $\beta = \sup\{i_x \mid x \in X\}$. Since $|X| = \kappa < \mu$ and μ is regular, $\beta < \mu$; hence $f_i < h_\beta$. $\square$

Definition 1.7. Let X be a set and I an ideal on X. A sequence $\vec{f}$ of functions in ${}^X\mathrm{Ord}$ of length λ is said to be **strongly I-increasing**, if there exist I-small sets $\langle Y_i \mid i < \lambda \rangle$ such that for all $i < j < \lambda$ and for all $x \in Y_i \cup Y_j$, $f_i(x) < f_j(x)$.

Theorem 1.8. *Let $|X| = \kappa$, let I be an ideal on X an $\vec{f}$ an $<_I$-increasing sequence. Then for any $\alpha \leq \mathrm{lh}(\vec{f})$ such that $\mathrm{cf}(\alpha) > \kappa$ the following are equivalent:*

1. *α is good.*
2. *There is a sequence $\langle h_i \mid i < \mathrm{cf}(\alpha) \rangle$ of functions in ${}^X\mathrm{Ord}$ which is pointwise increasing and cofinally interleaved with $\vec{f} \restriction \alpha$.*
3. *For every unbounded set $B \subseteq \alpha$ there is an unbounded set $A \subseteq B$ such that $\langle f_i \mid i \in A \rangle$ is strongly I-increasing.*
4. *There is an unbounded set $A \subseteq \alpha$ such that $\langle f_i \mid i \in A \rangle$ is strongly I-increasing.*

Proof.

(1. $\Rightarrow$ 2.) Construct $\vec{h}$ as in Remark 1.6 and use the same arguments.

(2. $\Rightarrow$ 3.) Suppose that $\vec{h} = \langle h_i \mid i < \mathrm{cf}(\alpha)\rangle$ is pointwise increasing and cofinally interleaved with $\vec{f} \restriction \alpha$. Let $B \subseteq \alpha$ be unbounded. For every $i < \mathrm{cf}(\alpha)$ we choose $\delta_i \in B$, $k_i, k_i' < \mathrm{cf}(\alpha)$ as follows:

- Let $\delta_0 \in B$ be the least ordinal in B such that there exists $k_0 < \mathrm{cf}(\alpha)$ such that $h_{k_0} <_I f_{\delta_0}$ and let $k_0' < \mathrm{cf}(\alpha)$ such that $f_{\delta_0} <_I h_{k_0'}$.
- If for all $j < i$, δ_j, k_j, k_j' are defined, then let $k_i > \sup_{j<i} k_j'$, $\delta_i \in B$ such that $h_{k_i} <_I f_{\delta_i}$ and $k_i' > k_i$ such that $f_{\delta_i} <_I h_{k_i'}$.

Then let $A = \{\delta_i \mid i < \mathrm{cf}(\alpha)\}$. Clearly, $A \subseteq B$ is unbounded. Now for every $\delta = \delta_i \in A$ we have

$$h_{k_i} <_I f_\delta <_I h_{k_i'}.$$

So for every $\delta = \delta_i \in A$ we can find $Y_\delta \in I$ such that for all $x \in X \setminus Y_\delta$, $h_{k_i}(x) < f_\delta(x) < h_{k_i'}(x)$. Then for $i < j < \mathrm{cf}(\alpha)$ and for $x \in X \setminus (Y_{\delta_i} \cup Y_{\delta_j})$ we have

$$f_{\delta_i}(x) < h_{k_i'}(x) < h_{k_j}(x) < f_{\delta_j}(x)$$

which proves the desired condition.

(3. $\Rightarrow$ 4.) is trivial.

(4. $\Rightarrow$ 1.) Assume that $A \subseteq \alpha$ is unbounded such that $\langle f_i \mid i \in A\rangle$ is strongly I-increasing and choose witnessing I-small sets Y_i for every $i \in A$. Without loss of generaliy assume that $\mathrm{otp}(A) = \mathrm{cf}(\alpha)$. Put

$$h(x) = \sup\{f_\delta(x) \mid \delta \in A \wedge x \in X \setminus Y_\delta\}.$$

Here we suppose without loss of generality that there is no x such that $x \in Y_\delta$ for every $\delta \in A$. We show that h is an eub for $\vec{f} \restriction \alpha$. Let $g <_I h$ and $Y \in I$ such that in $X \setminus Y$ $g < h$. Then for every $x \notin Y$ there is some $\delta_x \in A$ such that $x \notin Y_{\delta_x}$ and $g(x) < f_{\delta_x}(x)$. Now since $\mathrm{cf}(\alpha) > \kappa$ and A is unbounded there is $\delta \in A$ such that for all $x \notin Y$, $\delta_x < \delta$. Then for $x \notin (Y \cup Y_\delta)$, $g(x) < f_{\delta_x}(x) < f_\delta(x)$. Hence $g <_I f_\delta$.

Last but not least, we have to verify that for every $x \in X$, $\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$. Let

$$Y = \{x \in X \mid |\{\delta \in A \mid x \notin Y_\delta\}| < \mathrm{cf}(\alpha)\}.$$

Then for every $x \in Y$ there is $\delta_x \in A$ such that for every $\delta > \delta_x$, $x \in Y_\delta$. Since $\mathrm{cf}(\alpha) > \kappa$, there is $\delta \in A$ such that for every $x \in Y$, $\delta_x < \delta$. Then $Y \subseteq Y_\delta$ and hence Y is I-small. We can change the values of h on Y such that

$\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$ without affecting the property we have previously shown. On $X \setminus Y$, we clearly have $\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$. This proves 1.

$\square$

Definition 1.9. For $\delta < \lambda$ regular, define $S^\lambda_\delta = \{\eta \in \lambda \mid \mathrm{cf}(\eta) = \delta\}$.

Remark 1.10. For $\delta < \lambda$ regular infinite cardinals, $S^\lambda_\delta \subseteq \lambda$ is stationary.

Theorem 1.11. *Let $|X| = \kappa < \delta = \mathrm{cf}(\delta) < \lambda = \mathrm{cf}(\lambda)$, let I be an ideal on X and $\vec{f}$ be an I-increasing sequence such that $\mathrm{lh}(\vec{f}) = \lambda$. Then the following conditions are equivalent:*

1. *There are stationarily many good points in S^λ_δ.*
2. *There is an eub h for $\vec{f}$ such that for all $x \in X$, $\mathrm{cf}(h(x)) > \delta$.*

Proof.

(1. $\Rightarrow$ 2.) We will use the Trichotomy Theorem to show that we are in the Good case by checking that both the Bad and the Ugly case fail.

- Assume we are in the Ugly case. Then there is $g \in {}^X\mathrm{Ord}$ such that the sequence $\langle \{x \in X \mid f_i(x) < g(x)\} \mid i < \lambda\rangle$ is not eventually constant mod I. Equivalently, the sequence $\langle X_i \mid i < \lambda\rangle$, where $X_i = \{x \in X \mid g(x) \leq f_i(x)\}$ does not stabilize mod I. Then we can find a club $C \subseteq \lambda$ such that for every $i < j$ in C, $X_i \subsetneq_I X_j$. By our assumption, there is a good point $\alpha \in \mathrm{Lim}(C) \cap \mathrm{cof}(\delta)$. By the previous theorem, there is a sequence $\langle h_i \mid i < \delta\rangle$ which is pointwise increasing and cofinally interleaved with $\vec{f} \restriction \alpha$. Now since $\alpha \in \mathrm{Lim}(C)$, we can assume without loss of generality that for every $i < \delta$ there are $\beta < \gamma < \alpha$ in C such that

  $$h_i <_I f_\beta <_I f_\gamma <_I h_{i+1}. \tag{1}$$

  Now for $i < \delta$ let $Y_i = \{x \in X \mid g(x) \leq h_i(x)\}$. By (1) we obtain that

  $$Y_i \subseteq_I X_\beta \subsetneq_I X_\gamma \subseteq_I Y_{i+1}. \tag{2}$$

  In particular, (2) implies that for any $i < j < \delta$, $Y_i \subsetneq_I Y_j$, thus the sequence $\langle Y_i \mid i < \delta\rangle$ is not eventually constant mod I. This contradicts the fact that $\vec{h}$ has an eub.

- Assume now that we are in the Bad case. Take sets $S_x \subseteq \mathrm{Ord}$ for $x \in X$ such that $|S_x| < \delta$ and an ultrafilter U on X such that $U \cap I = \emptyset$ and for every $\alpha < \lambda$ there are $g \in \prod_{x\in X} S_x$ and $\beta < \lambda$ such that $f_\alpha <_U g <_U f_\beta$. Then there is a club set $C \subseteq \lambda$ such that for every $\alpha < \beta$ in C there is $g \in \prod_{x\in X} S_x$ with $f_\alpha <_U g <_U f_\beta$. Choose a good point $\alpha \in \mathrm{Lim}(C) \cap \mathrm{cof}(\delta)$ and an eub h for $\vec{f} \upharpoonright \alpha$ such that for every $x \in X$, $\mathrm{cf}(h(x)) = \delta$. Now define $g(x) = \sup S_x \cap h(x)$. Since $\mathrm{cf}(g(x)) < \delta$ we obtain that $g < h$, thus there is $\beta < \alpha$ such that $g <_I f_\beta$. But the Bad case allows us to find $g' \in \prod_{x\in X} S_x$ and $\gamma < \alpha$ with $f_\beta <_U g' <_U f_\gamma$. Since $U \cap I = \emptyset$, this yields

  $$g <_U f_\beta <_U g' <_U f_\gamma <_U h.$$

  which clearly contradicts the definition of g.

Since we are in the Good case, there is an eub h for $\vec{f}$ such that $\mathrm{cf}(h(x)) \geq \delta$ for every $x \in X$. Let $Y = \{x \in X \mid \mathrm{cf}(h(x)) = \delta\}$. Suppose for a contradiction that $Y \in I^+$. Then consider the ideal $J = \{Z \cap Y \mid Z \in I\}$ on Y. Since $Y \in I^+$, $\bar{h} = h \upharpoonright Y$ is still an eub for $\langle f_i \upharpoonright Y \mid i < \lambda\rangle$. But then Remark 1.6 implies that $\lambda = \delta$ which is absurd. By modifying h on an I-small set, we obtain 2.

(2. $\Rightarrow$ 1.) Let h be an eub for $\vec{f}$ such that $\mathrm{cf}(h(x)) = \delta$ for every $x \in X$. Without loss of generality we assume that $f_i < h$ for every $i < \lambda$. Put

$$S = \{\alpha \in S^\lambda_\delta \mid \alpha \text{ is a good point}\}.$$

Let $C \subseteq \lambda$ be a club set. Inductively, we construct an increasing sequence $\langle \alpha_i \mid i < \delta\rangle$ of ordinals in C as follows: At stage $i < \delta$ let $g_i(x) = \sup\{f_{\alpha_j}(x) \mid j < i\}$ for every $x \in X$. Now since $f_{\alpha_j}(x) < h(x)$ for every $x \in X$ and every $j < i$ and $\mathrm{cf}(h(x)) > \delta$, we get that $g_i(x) < h(x)$ for all $x \in X$. Hence there is $\alpha_i \in C \setminus \bigcup_{j<i}(\alpha_j + 1)$ such that $g_i <_I f_{\alpha_i}$.

Now let $\alpha = \sup\{\alpha_i \mid i < \delta\} \in S^\lambda_\delta \cap C$. We show that $\alpha \in S$. Clearly, $\vec{f} \upharpoonright \alpha$ is cofinally interleaved with $\vec{g} = \langle g_i \mid i < \delta\rangle$ and since $\vec{g}$ is $\leq$-increasing, $g = \sup\{g_i \mid i < \delta\}$ is an eub for $\vec{g}$ and by Remark 1.2 also for $\vec{f} \upharpoonright \alpha$.

$\square$

Definition 1.12. Let $S \subseteq \lambda$ be a stationary set. Then a sequence $\langle C_\eta \mid \eta \in S\rangle$ is said to be a **club guessing sequence**, if it satisfies

1. $C_\eta \subseteq \eta$ is club for every $\eta \in S$,

2. For every club $E \subseteq \lambda$ there is $\eta \in S$ such that $C_\eta \subseteq E$.

Theorem 1.13 (Shelah's Club Guessing Theorem)**.** *Let δ, λ be regular cardinals such that $\delta^+ < \lambda$ and let $S \subseteq S^\lambda_\delta$ be stationary. Then S has a club guessing sequence $\langle C_\eta \mid \eta \in S\rangle$ such that $\operatorname{otp}(C_\eta) = \delta$ for every $\eta \in S$.*

Proof. Since we only need the case where $\delta > \aleph_0$, we will only prove this case. We start with any sequence $\vec{C} = \langle C_\eta \mid \eta \in S\rangle$ of club sets $C_\eta \subseteq \eta$ of order type δ. For a club $E \subseteq \lambda$ put

$$\vec{C} \restriction E = \langle C_\eta \cap E \mid \eta \in S \cap \operatorname{Lim}(E)\rangle.$$

Observe that if $\eta \in S \cap \operatorname{Lim}(E)$, then $C_\eta \cap E \subseteq \eta$ is club. We will construct some $E \subseteq \lambda$ club such that $\vec{C} \restriction E$ is a club guessing sequence for $S \cap \operatorname{Lim}(E)$. If such E exists, then we extend it to a club guessing sequence for S by taking C_η for any $\eta \in S \setminus \operatorname{Lim}(E)$.

Now suppose for a contradiction that for every $E \subseteq \lambda$ club there exists club $D_E \subseteq \lambda$ such that for every $\eta \in S \cap \operatorname{Lim}(E)$, $C_\eta \cap E \nsubseteq D_E$. Inductively, construct a sequence $\langle E_\xi \mid \xi < \delta^+\rangle$ of club subsets of λ as follows:

- $E_0 = \lambda$.

- If $\xi < \delta^+$ is a limit ordinal and for every $i < \xi$, E_i is defined, let $E_\xi = \bigcap_{i<\xi} E_i$. Since $\xi < \lambda$, $E_\xi \subseteq \lambda$ is club.

- Given E_ξ, define $E_{\xi+1} = \operatorname{Lim}(E_\xi \cap D_{E_\xi})$.

By construction we obtain that for any $\eta \in S \cap \operatorname{Lim}(E_\xi)$, $C_\eta \cap E_\xi \nsubseteq E_{\xi+1}$. Finally, define $E = \bigcap_{\xi<\delta^+} E_\xi$. Since $\delta^+ < \lambda$, E is a club subset of λ. Now consider some $\eta \in S \cap E$. Then the sequence $\langle C_\eta \cap E_\xi \mid \xi < \delta^+\rangle$ is decreasing with respect to $\subseteq$, but since every $|C_\eta \cap E_\xi| = \delta$ for every $\xi < \delta^+$ it must stabilize, i.e. there is $\xi < \delta^+$ such that for every $\zeta > \xi$, $C_\eta \cap E_\zeta = C_\eta \cap E_\xi$. But $\eta \in S \cap E_{\xi+1}$, hence $C_\eta \cap E_\xi \nsubseteq E_{\xi+1}$ which is absurd. $\square$

Theorem 1.14 (Abraham, Magidor)**.** *Let $|X| = \kappa$, I an ideal on X and $\kappa < \delta < \delta^+ < \rho < \lambda$ be regular cardinals. Let $\vec{f}$ be an I-increasing sequence such that $\operatorname{lh}(\vec{f}) = \lambda$ and assume that for every $\xi \in S^\lambda_\rho$ there is a club $E_\xi \subseteq \xi$ such that $\sup_{i \in E_\xi} f_i \leq_I f_\xi$. Then $\vec{f}$ has an eub h such that $\operatorname{cf}(h(x)) > \delta$ for all $x \in X$.*

Proof. By Theorem 1.11 it is enough to show that

$$S = \{\alpha \in S^\lambda_\delta \mid \alpha \text{ is a good pooint}\}$$

is stationary in λ.

Let $E \subseteq \lambda$ be club. Since $\delta^+ < \rho$, there is a club guessing sequence $\langle C_\eta \mid \eta \in S^\rho_\delta \rangle$ for S^ρ_δ. We construct an increasing sequence $\bar{\gamma} = \langle \gamma_i \mid i < \rho + 1 \rangle$ of elements of E as follows:

- Let $\gamma_0 \in E$.
- Suppose that $\gamma_i \in E$ has already been defined. For every $\eta \in S^\rho_\delta$ let $h^i_\eta(x) = \sup\{f_{\gamma_j}(x) \mid j \in C_\eta \cap (i+1)\}$; furthermore, let $i_\eta < \lambda$ such that $h^i_\eta <_I f_{i_\eta}$, if such i_η exists; else $i_\eta = \gamma_i + 1$. Since $\rho < \lambda$ and E is club, there is $\gamma_{i+1} \in E$ such that $\gamma_{i+1} \geq \sup(\{i_\eta \mid \eta \in S^\rho_\delta\} \cup \{\gamma_i + 1\})$.
- In the limit case, define $\gamma_i = \sup_{j<i} \gamma_j \in E$.

Now put $D = \{i < \rho \mid \gamma_i \in E_{\gamma_\rho}\}$.

Claim. D is closed unbounded in ρ.

Proof. D is closed since E_{γ_ρ} is closed. Clearly, the set $C = \{\gamma_i \mid i < \rho\}$ is closed unbounded in γ_ρ and $\gamma_\rho \in S^\lambda_\rho$. Hence $C \cap E_{\gamma_\rho}$ is club in γ_ρ. But this implies that D is club in ρ. □

Now pick $\eta \in S^\rho_\delta$ such that $C_\eta \subseteq D$.

Claim. The sequence $\langle f_{\gamma_i} \mid i \in C_\eta \setminus \mathrm{Lim}(C_\eta) \rangle$ is strongly increasing.

Proof. For every $i \in C_\eta$, $\gamma_i \in E_{\gamma_\rho}$ which means that $f_{\gamma_i} \leq_I f_{\gamma_\rho}$. By construction of γ_i, this means that $h^i_\eta <_I f_{\gamma_{i+1}} \leq_I f_{\gamma_j}$, where $j = \min\{k \in C_\eta \mid k > i\}$ is the C_η-successor of i. Now let $Y_j \in I$ witness this. Now every $i \in C_\eta \setminus \mathrm{Lim}(C_\eta)$, has a predecessor in C_η. Consider $i \leq k < j$ in $C_\eta \setminus \mathrm{Lim}(C_\eta)$ such that k is the predecessor of j in C_η. Then for every $x \in X \setminus Y_j$

$$f_{\gamma_i}(x) \leq h^k_\eta(x) < f_{\gamma_j}(x).$$

□

Since $\{\gamma_i \mid i \in C_\eta \setminus \mathrm{Lim}(C_\eta)\}$ is unbounded in γ_η, Theorem 1.8 implies that γ_η is a good point and $\mathrm{cf}(\gamma_\eta) = \mathrm{cf}(\eta) = \delta$. Hence $\gamma_\eta \in S \cap E$. □

2 Applications

Recall our goal to prove the following result of Shelah:

Theorem 2.1 (Shelah)**.** *There is an infinite set $A \subseteq \omega$ and a sequence $\vec{f} = \langle f_i \mid i < \aleph_{\omega+1}\rangle$ such that $\vec{f}$ is increasing and cofinal in $\prod_{n\in A} \aleph_n$ under the eventual domination ordering.*

We have already reduced the proof of this result to

Lemma 2.2. *There is a sequence $\vec{f} = \langle f_i \mid i < \aleph_{\omega+1}\rangle$ and a function h with $f_i \in \prod_{n\in\omega} \aleph_n$ and $h \in \prod_{n\in\omega}(\aleph_n + 1)$ such that the following properties hold:*

1. *The sequence $\vec{f}$ is increasing and cofinal in $\prod_{n\in\omega} h(n)$ under the eventual domination ordering.*
2. *For each $m \in \omega$, the set $B_m = \{n \in \omega \mid \mathrm{cf}(h(n)) = \aleph_m\}$ is finite.*

Proof. We are in the case $|X| = \omega$ and $I = \{Y \subseteq \omega \mid |Y| < \aleph_0\}$ and we want to find an I-increasing sequence $\vec{f} \in \prod_{n\in\omega} \aleph_n$ and an eub $h \in \prod_{n\in\omega} \aleph_{n+1}$. We build $\vec{f}$ inductively as follows:

- Let f_0 be any function in $\prod_{n\in\omega} \aleph_n$.
- In the successor case, let $f_{i+1}(n) = f_i(n) + 1$.
- If $\alpha < \aleph_{\omega+1}$ is a limit, let $\delta = \mathrm{cf}(\alpha) < \aleph_\omega$. Now let $E_\alpha \subseteq \alpha$ be closed unbounded of order type $\mathrm{cf}(\alpha)$. Let $\delta = \aleph_m$ for some $m \in \omega$, then put

$$f_\alpha(n) = \begin{cases} \sup\{f_i(n) \mid i \in E_\alpha\}, & n > m \\ 0 & \text{else.} \end{cases}$$

  By construction, since $\mathrm{otp}(E_\alpha) = \aleph_m$, for $n > m$ we have $\sup\{f_i(n) \mid i \in E_\alpha\} < \aleph_n$. Thus $f_\alpha \in \prod_{n\in\omega} \aleph_n$.

Claim. The sequence $\vec{f} = \langle f_i \mid i < \aleph_{\omega+1}\rangle$ is I-increasing.

Proof. Let $\alpha < \aleph_{\omega+1}$ and assume that for every $\beta < \alpha$ and for every $i < \beta$, $f_i <_I f_\beta$. Let $i < \alpha$. We check that $f_i <_I f_\alpha$. If α is a successor ordinal, this is trivial. Suppose that α is a limit and choose $\beta \in E_\alpha$ such that $i < \beta$. Then by construction $f_\beta <_I f_\alpha$ and by induction hypothesis also $f_i <_I f_\beta$. □

Now we are ready to apply Theorem 1.14. For any $n > 0$ in ω, we obtain an eub h_n for $\vec{f}$ such that $\mathrm{cf}(h_n(x)) > \aleph_n$ for all $x \in X$.

Claim. $h = h_1$ already satisfies the desired properties.

Proof. Assume that $m > 1$ such that B_m is infinite. Since both h and h_m are eubs, we have $h =_I h_m$. But $\mathrm{cf}(h_m(x)) > \aleph_m$ for every $x \in X$. Contradiction. □

□

Theorem 2.3. *Let μ be a singular cardinal with $\mathrm{cf}(\mu) = \kappa > \omega$ and let $C \subseteq \mu$ be a club set of singular cardinals. Then there is a club subset $D \subseteq C$ and a sequence $\vec{f}$ of length μ^+ which is increasing and cofinal in $\prod_{\lambda \in D} \lambda^+$ modulo the nonstationary ideal NS.*

Proof. Like in the previous theorem, we construct a sequence $\vec{f} \in \prod_{\lambda \in C} \lambda^+$ with an eub h such that for every $\delta < \mu$, $\{\lambda \in C \mid \mathrm{cf}(h(\lambda)) = \delta\}$ is non-stationary. We need to verify that $h(\lambda) = \lambda^+$ for NS-almost all $\lambda \in C$. If not, then the set

$$S = \{\lambda \in C \mid \mathrm{cf}(h(\lambda)) < \lambda\}$$

is stationary in μ; moreover, on S the function $\lambda \mapsto \mathrm{cf}(h(\lambda))$ is regressive and hence by Fodor there is some $\delta \in \mu$ such that $\{\lambda \in C \mid \mathrm{cf}(h(\lambda)) = \delta\}$ is stationary. Contradiction. □

Theorem 2.4 (Silver)**.** *Let κ be a singular cardinal such that $\mathrm{cf}(\kappa)$ is uncountable. Suppose that there is a stationary set $S \subseteq \kappa$ such that for every $\delta \in S$, $\delta^{\mathrm{cf}(\kappa)} = \delta^+$. Then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.*

Proof. Let $S \subseteq \kappa$ be a stationary set of order type $\mathrm{cf}(\kappa)$ such that for all $\delta \in S$ we have $\delta^{\mathrm{cf}(\kappa)} = \delta^+$. By the previous theorem, there is a club set $C \subseteq \kappa$ of singular cardinals and a sequence $\bar{f} = \langle f_i \mid i < \kappa^+\rangle$ which is NS-increasing and cofinal in $\prod_{\delta \in C} \delta^+$. Now by replacing S by $S \cap C$ and f_ξ by $f_\xi \restriction S$ we obtain that $\bar{f}$ is NS-increasing and cofinal in $\prod_{\delta \in S} \delta^+$.

Since every $\delta \in S$ satisfies $\delta^{\leq \mathrm{cf}(\kappa)} = \delta^+$, we can define bijective maps

$$c_\delta : [\delta]^{\leq \mathrm{cf}(\kappa)} \to \delta^+$$

We use this to code sets $X \in [\kappa]^{\mathrm{cf}(\kappa)}$ by

$$h_X \in \prod_{\delta \in S} \delta^+, \delta \mapsto c_\delta(X \cap \delta).$$

Claim. If $X \neq Y$, then there is $\delta \in S$ such that for all $\lambda \geq \delta$, $h_X(\lambda) \neq h_Y(\lambda)$.

Proof. If $X \neq Y$, then there is a minimal $\delta \in S$ such that $X \cap \delta \neq Y \cap \delta$. Then for every $\lambda \geq \delta$, $X \cap \lambda \neq Y \cap \lambda$ and hence $h_X(\lambda) = c_\lambda(X \cap \lambda) \neq c_\lambda(Y \cap \lambda) = h_Y(\lambda)$. □

Claim. For every $g \in \prod_{\delta \in S} \delta^+$, the set

$$F_g = \{X \in [\kappa]^{\mathrm{cf}(\kappa)} \mid h_X <_{\mathrm{NS}} g\}$$

has cardinality $\leq \kappa$.

Proof. Suppose that for some g, $|F_g| \geq \kappa^+$. For $\delta \in S$, $g(\delta) < \delta^+$, so there is an enumeration $\langle g_i^\delta \mid i < i_\delta \rangle$ of order type $i_\delta \leq \delta$. Now define for $X \in F_g$, $S_X = \{\delta \in X \mid h_X(\delta) < g(\delta)$ and a function k_X on S_X by $k_X(\delta) = i$ for the unique $i < i_\delta$ such that $h_X(\delta) = g_i^\delta$. Then S_X is stationary in κ and k_X is regressive, so there is a stationary $T_X \subseteq S_X$ and $\delta_X < \kappa$ such that $k_X(\delta) < \delta_X$ for all $\delta \in T_X$. Since there are at most $2^{\mathrm{cf}(\kappa)}$ subsets of S and $2^{\mathrm{cf}(\kappa)} < \kappa^+ \leq |F_g|$, there is $F_0 \subseteq F_g$ such that $|F_0| = \kappa^+$, a stationary set $S_0 \subseteq S$ and $\delta_0 < \kappa$ such that for all $X \in F_0$, $S_X = S_0$ and $\delta_X = \delta_0$. Without loss of generality, assume that $\delta_0 \in S$. Now there are at most $\delta_0^{\mathrm{cf}(\kappa)} = \delta_0^+$ many different possibilities for the functions $k_X, X \in F_0$. So (by thinning out F_0) we can also assume that for all $X \in F_0$ the functions k_X are all the same k_0. But then for $X \in F$ and $\delta \in S_0$ we have $h_X(\delta) = g^\delta_{k_X(\delta)} = g^\delta_{k_0(\delta)}$ which does not depend on X. This contradicts the previous claim. □

Now we obtain that for every $g \in \prod_{\delta \in S} \delta^+$, $|F_g| \leq \kappa$. But since $\vec{f}$ is cofinal in $\prod_{\delta \in S} \delta^+$, every $X \in [\kappa]^{\mathrm{cf}(\kappa)}$ is in some F_ξ, $\xi < \kappa^+$. This implies

$$\kappa^{\mathrm{cf}(\kappa)} = |[\kappa]^{\mathrm{cf}(\kappa)}| \leq |\bigcup_{\xi < \kappa^+} F_\xi| \leq \kappa^+ \cdot \kappa = \kappa^+.$$

□

Corollary 2.5 (Silver)**.** *Let κ be a singular cardinal of uncountable cardinality and assume that* GCH *holds below κ. Then $2^\kappa = \kappa^+$.*

Proof. First observe that the set $S = \{\delta < \kappa \mid \mathrm{cf}(\delta) < \mathrm{cf}(\kappa) < \delta\}$ is stationary in κ. Now for every $\delta \in S$ we have

- $\delta < \delta^{\mathrm{cf}(\delta)} \leq \delta^{\mathrm{cf}(\kappa)}$
- $\delta^{\mathrm{cf}(\kappa)} \leq \delta^\delta = 2^\delta = \delta^+$.

This clearly implies that if $\delta \in S$, then $\delta^{\mathrm{cf}(\kappa)} = 2^\delta = \delta^+$. Thus the previous theorem implies that $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$ and thus in particular $2^\kappa = \kappa^+$ since for $\langle \kappa_i \mid i < \mathrm{cf}(\kappa) \rangle$ cofinal in κ,

$$2^\kappa = 2^{\sum_{i<\mathrm{cf}(\kappa)} \kappa_i} = \prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i} \leq \kappa^{\mathrm{cf}(\kappa)}.$$

$\square$