Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 3

Problem 10 (4 Points)**.**

(1) Show that (x, y), $x \times y$ and $f \upharpoonright x$ are definite terms.
(2) Show that $\mathrm{TC}(x)$ is a definite term.
(3) Show that the term V_ω is definite.

Problem 11 (*Richard's paradox: The undefinability of definability*, 4 Points)**.** Show that there is no $\in$-formula $\varphi(v_0, v_1)$ with

$$\mathrm{ZFC} \vdash \forall n \in \mathrm{Fml}\ \forall x\ \forall y\ [(\varphi(n,x) \wedge \varphi(n,y)) \longrightarrow x = y]$$

and

$$\mathrm{ZFC} \vdash \exists x\ \forall y\ [\psi(y) \leftrightarrow x = y) \longrightarrow \forall y\ (\varphi(\ulcorner \psi \urcorner, y) \leftrightarrow \psi(y))]$$

for every $\in$-formula $\psi(v)$ (Hint: *Show that there is an ordinal* α *with* $\neg\varphi(n, \alpha)$ *for all* $n \in \mathrm{Fml}$ *and consider the least ordinal with this property*).

Problem 12 (4 Points)**.** Assume ZF. The following statements are equivalent.

(1) $\mathrm{V} = \mathrm{HOD}$.
(2) There is a well-ordering of V of order-type Ord that is definable by a formula without parameters.

Problem 13 (8 Points)**.** Extend the formal language Fml to a language Fml′ by adding atomic formulas for "$v_i \in A$" where A is considered a unary predicate symbol. One could, e.g., code $v_i \in A$ by $(5, i, i)$. For a structure (M, A, E) with $M \in \mathrm{V}$, $A \subseteq M$, $\varphi \in \mathrm{Fml}'$, and a an assignment in M define the satisfaction relation $(M, A, E) \models \varphi[a]$ recursively as in Definition 16. Define

$$\mathrm{OD}(A) \ = \ \{y \mid \exists \alpha \in \mathrm{Ord}\ \exists \varphi \in \mathrm{Fml}'\ \exists a \in \mathrm{Asn}((A \cup \alpha) \cap \mathrm{V}_\alpha) \\ y = \{z \in \mathrm{V}_\alpha \mid (\mathrm{V}_\alpha, A \cap \mathrm{V}_\alpha, \in) \models \varphi[a\tfrac{z}{0}]\}\}$$

and the corresponding generalization $\mathrm{HOD}(A)$ of HOD. Prove:

(1) If A is transitive, then $A \subseteq \mathrm{HOD}(A)$.
(2) If A is moreover definable from some parameters $a_0, \ldots, a_{n-1} \in A$, then $\mathrm{ZF}^{\mathrm{HOD}(A)}$.

Please drop your solutions into Box 6 until Thursday, May 02, 10:00 a.m.